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Scalar field corrections to AdS$_4$ gravity from higher spin gauge theory

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Abstract: We compute the complete contribution to the stress-energy tensor in the minimal bosonic higher spin theory in $D = 4$ that is quadratic in the scalar field. We find arbitrarily high derivative terms, and that the total sign of the stress-energy tensor depends on the parity of the scalar field.

Keywords: AdS-CFT and dS-CFT Correspondence, Gauge Symmetry.

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1. Introduction

By now there is plenty of evidence for the remarkable correspondence between field theories in d dimensions and string/M theory on $AdS_{d+1} \times M$ spacetimes. In the conformal cases with maximal number of supersymmetries the correspondence relates the low-energy limits of two complementary descriptions of the sector of the theory with N units of D3-brane or M2/5-brane charge [1]. To be more precise, the correspondence relates the $1/N$ expansions of the generating functionals on both sides. This is a remarkable relation, in the sense that on the CFT side N is, roughly, the number of colours of the sigma-model living on the stack of branes, while on the bulk side $1/N$ plays the role of Planck's constant. A crucial property of the $1/N$ expansion of the CFT is that its correlators factorise in the limit $N \to \infty$, such that it makes sense to identify the connected part of the correlators with the connected Feynman diagrams of the bulk theory with external bulk-to-boundary propagators.

In fact, starting from any CFT in d dimensions that factorises in some limit and that has a well-defined generating functional (which is a quite non-trivial condition), it should be possible to reconstruct an effective action in $d+1$ dimensions which has an AdS vacuum and which reproduces the correlators as described above. Moreover, within this context there is a correspondence between the global, continuous symmetries of the CFT and the local symmetries of the bulk theory. In particular, this implies that the bulk theory necessarily contains gravity. It is then rather gratifying from a string theorist's point of view that AdS/CFT correspondence arises naturally within string/M theory.

The most studied example relates $d = 4$, $\mathcal{N} = 4$ Yang-Mills theory with SU(N) gauge group to the Type IIB string theory on $AdS_5 \times S^5$ of radius R with string coupling g_s, string tension T_s and N units of five-form flux, which is an exact solution to the string theory provided that $R^2 T_s = \sqrt{4\pi g_s N}$. The bulk string theory is notoriously difficult to quantise starting from the worldsheet formulation. This correspondence has therefore been tested primarily by comparing the weak coupling limit of the worldsheet theory, i.e. the supergravity limit, to strong coupling results in SYM obtained either by studying correlators protected by some symmetries or by summing up the four-dimensional perturbation series for weak 't Hooft coupling $\lambda = N g^2_{\mathrm{YM}} \ll 1$ and continuing the result to $\lambda \sim R^4 T_s^2 \gg 1$.

However, as argued above, it should be possible to test the correspondence directly order by order in the $1/N$ expansion on both sides. In particular, it is interesting to consider the free limit $\lambda \to 0$ of the SYM theory. The generating functional of composite SU(N) invariant operators remains highly non-trivial in this limit and has two remarkable properties: (i) it admits a consistent truncation to the generating functional of bilinear operators (which is important since there is no mass-gap); (ii) the primary bilinear operators are superfields which contain conserved currents of arbitrarily high spin. Hence, the free limit of SYM corresponds to a limit of the Type IIB theory in which it develops higher spin gauge symmetry and admits a consistent truncation to the massless sector. These features should be sufficient to determine the effective five-dimensional bulk action in the massless sector up to some number of interaction ambiguities. The first steps towards this

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have been taken in [2]–[6], where the precise higher spin algebra, the massless spectrum and linearised field equations have been given, and in [7] where certain cubic interactions have been constructed.

Massless higher spin theories are further developed in four dimensions, where the full field equations are known [8]–[11]. The unbroken phase of the higher spin gauge theory corresponds to the generating functional of bilinear operators in free singleton field theory. By various deformations which breaks the higher spin symmetry while preserving some colour symmetry group one can flow to strongly coupled interacting fixed points with some $1/N$ expansion. In the case of 32 supersymmetries in the bulk, by considering $\mathrm{SU}(N)$ colour symmetry and extending the bulk theory with massive fields and the generating functional of the field theory with multi-linear single trace operators the theory has been conjectured to flow to the IR fixed point of the Yang-Mills theory dual to 11 dimensional M-theory/supergravity in the bulk. This flow is analogous to the $\lambda \to \infty$ in the Type IIB case.

Another interesting deformation, which preserves maximal $\mathrm{O}(N)$ colour symmetry, is generated by perturbing the free non-supersymmetric scalar singleton theory with a double trace operator fJ^2, where $J = \varphi^a\varphi^a$ is the scalar operator and f is a parameter of dimension energy, which results in a flow to the interacting fixed point of the $\mathrm{O}(N)$ model (in the limit of large f in units of some fixed length [12]). It was recently proposed in [13] that this flow has a dual description in terms of the minimal bosonic higher spin gauge theory in four dimensions (with parity even scalar field), such that the two fixed points correspond to the $\Delta_\pm$ boundary conditions of the scalar field. This proposal has been investigated further in [14, 16].

The simplest way of testing the correspondence is to compare free field theory correlation functions with the bulk amplitudes of the higher spin gauge theory in the vacuum with Δ_- boundary condition [4, 17, 18]. The closed form of the full field equations involves many auxiliary fields. The physical field equations can be obtained by eliminating the auxiliary fields order by order in a curvature expansion scheme. No action reproducing any of these forms of the field equations is known, however. In this paper we take the first steps towards finding the cubic action for a subset of the physical fields.

As we will see in the introductory section 2, the minimal bosonic higher spin theory in AdS_4 is based on an algebra which is a minimal extension of the AdS_4 algebra, here called $hs(4)$. A unitary irreducible representation of $hs(4)$ can be constructed from the symmetric product of two spin 0 singletons, and consists of massless fields of spins $s = 0, 2, 4, \ldots$. The scalar, which we denote by ϕ, is even under parity and its ground state has AdS energy $E_0 = 1$. This model can be obtained by truncating various supersymmetric theories [3, 11].

In this paper we expand the minimal bosonic higher spin gauge theory by treating the scalar field ϕ and the higher spin fields as well as all curvatures as weak fields, and calculate contributions to the Einstein equation from the scalar ϕ. In the leading order $s = 2$ field equation coincides with the Einstein equation with a cosmological constant. The first non-trivial corrections from the scalar field are contributions to the stress-energy tensor according to

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \mathrm{Re}\{b_1^2\}\Bigg[\sum_k \frac{2^k}{(k!)^2}\Bigg(\xi(k) g_{\mu\nu}\nabla_{\rho\{k+1\}}\phi\ \nabla^{\rho\{k+1\}}\phi + \\ + \eta(k)\nabla_{\rho\{k\}\mu}\phi\nabla^{\rho\{k\}}{}_{\nu}\phi + \\ + \zeta(k)\nabla_{\rho\{k\}\mu\nu}\phi\ \nabla^{\rho\{k\}}\phi\Bigg) - \frac{4}{9}g_{\mu\nu}\,\phi\,\phi\Bigg], \tag{1.1}$$

where

$$\nabla_{\mu\{k\}} \equiv \nabla_{(\mu_1}\ldots\nabla_{\mu_k)} - \mathrm{traces}\,, \qquad \nabla_{\mu\{k\}\nu} \equiv \nabla_{(\mu_1}\ldots\nabla_{\mu_k}\nabla_{\nu)} - \mathrm{traces}\,, \tag{1.2}$$

and b_1 is a complex constant which enters the closed form of the field equations. The functions ξ, η and ζ are shown explicitly in eqs. (3.22)–(3.24). There are two qualitatively interesting properties, namely the higher derivative nature of the stress energy tensor, and the potential sign ambiguity in $\mathrm{Re}\{b_1^2\}$, which we shall discuss further in the last section. We shall also discuss what further computations needs to be done in order to find the cubic action in the spin $s = 0, 2$ sector.

The papers is organised as follows. In section 2 we briefly review the higher spin theory in AdS_4. In section 3 we use the weak field expansion scheme to work out the second order scalar corrections to the spin $s = 2$ field equation. Section 4 concludes with a summary and a discussion about the relevance of the result. Details on the actual calculation can be found in the appendices A–D. We have made an effort to present the calculation in such a way that the interested reader may follow it step by step.

2. Higher spin formalism in $D = 4$

Below we review the framework of higher spin theory in $D = 4$. It is instructive to first examine pure gravity, then extending to the field content of the minimal bosonic higher spin gauge theory, which in particular contains the scalar field whose stress energy tensor will be examined in the next section.

2.1 Pure AdS_4 Gravity

We start by formulating gravity with a negative cosmological constant as a constrained system of 0-forms and 1-forms based on the AdS algebra $SO(3, 2)$,

$$\begin{aligned} [M_{ab}, M_{cd}] &= i\eta_{bc}M_{ad} - i\eta_{bd}M_{ac} + i\eta_{ad}M_{bc} - i\eta_{ac}M_{bd}\,, \\ [M_{ab}, P_c] &= i\eta_{bc}P_a - i\eta_{ac}P_b\,, \\ [P_a, P_b] &= iM_{ab}\,. \end{aligned} \tag{2.1}$$

Out of the generators above we can define an $SO(3, 2)$-valued connection 1-form E as

$$E \equiv -i\Big(e^a P_a + \frac{1}{2}\omega^{ab}M_{ab}\Big) \equiv e + \omega = -E^\dagger\,, \tag{2.2}$$

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where e^a is the vierbein and ω^{ab} is the Lorentz connection. The factor of $-i$ is inserted for later convenience. The field strength of E becomes

$$\mathcal{R} \equiv dE + E \wedge E = \mathcal{R}^a P_a + \frac{1}{2}\mathcal{R}^{ab} M_{ab}\,, \tag{2.3}$$

where

$$\begin{aligned} \mathcal{R}^a &= -i(de^a + \omega^a{}_c \wedge e^c) = -iT^a\,, \\ \mathcal{R}^{ab} &= -i(d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b + e^a \wedge e^b) = -i(R^{ab} + e^a \wedge e^b)\,, \end{aligned} \tag{2.4}$$

where T^a is the torsion and R^{ab} the Riemann tensor. The equations for gravity follow from the following curvature constraints:

$$\mathcal{R}^a = 0\,, \tag{2.5}$$

$$\mathcal{R}^{ab} = -i\, e_c \wedge e_d \phi^{abcd}\,, \tag{2.6}$$

where ϕ^{abcd} is the Weyl tensor, which belongs to the $\yng(2,2)$ representation of SO(3,1). The Weyl tensor describes the traceless part of $R_{\mu\nu}{}^{ab}$. Eq. (2.5) fixes the Lorentz connection ω^{ab} in terms of the vierbein e^a and the trace of (2.6) gives the Einstein equation,

$$\mathrm{Ric}_\mu{}^a - \frac{1}{2}Re_\mu{}^a + \Lambda e_\mu{}^a = 0\,, \tag{2.7}$$

with a negative cosmological constant $\Lambda = -3$.

From the curvature identities we conclude that the Weyl tensor must obey

$$\nabla\phi^{abcd} = e_f \phi^{abcdf}\,, \tag{2.8}$$

$$\nabla\phi^{abcdf} = e_g \phi^{abcdfg} + 4e^f \phi^{abcd} + 5e_g \phi^{abch} \phi_h{}^{dfg} \tag{2.9}$$

$$\vdots$$

where ϕ^{abcdf} is $\yng(3,2)$, ϕ^{abcdfg} is $\yng(4,2)$, and so on. The Bianchi identity $\nabla R_{ab} = 0$ holds since (2.8) implies that $\nabla^f \phi^{abcd}$ is $\yng(3,2)$. The integrability of (2.8) implies (2.9) and so forth. Eqs. (2.5) and (2.6) together with (2.8) and (2.9) are invariant under SO(3,2) gauge transformations, under which e^a and ω^{ab} transforms as gauge fields in the adjoint representation. The Weyl tensor ϕ^{abcd} and all its derivatives form another, infinite dimensional representation, which generalises from spin two to arbitrary spin, as we shall discuss below.

2.2 Generalisation to higher spin gauge theory

The framework outlined in the previous paragraph is suitable for formulating gauge theories based on higher spin extensions of SO(3,2) [9]. To facilitate these constructions one introduces Grassmann even oscillators y_α and $\bar{y}_{\dot\alpha} = (y_\alpha)^\dagger$, which are Weyl spinors and obey the following algebra,

$$\begin{aligned} y_\alpha \star y_\beta &= y_\alpha y_\beta + i\epsilon_{\alpha\beta}\,, & \bar{y}_{\dot\alpha} \star \bar{y}_{\dot\beta} &= \bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta} \\ y_\alpha \star \bar{y}_{\dot\beta} &= y_\alpha \bar{y}_{\dot\beta} & \bar{y}_{\dot\alpha} \star y_\beta &= \bar{y}_{\dot\alpha} y_\beta\,, \end{aligned} \tag{2.10}$$

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where $\star$ denotes the associative product of oscillators and the products on the right hand sides, written without stars, are Weyl ordered (our spinor conventions are given in appendix A). This extends to general Weyl ordered functions as

$$f(y,\bar{y})\star g(y,\bar{y}) = f(y,\bar{y})e^{-i(\overleftarrow{\partial}^{\alpha}\overrightarrow{\partial}_{\alpha}+\overleftarrow{\partial}^{\dot{\alpha}}\overrightarrow{\partial}_{\dot{\alpha}})}g(y,\bar{y})\,, \tag{2.11}$$

where $\partial_\alpha \equiv \partial/\partial y^\alpha$ and $\partial^\alpha \equiv \epsilon^{\alpha\beta}\partial_\beta = -\partial/\partial y_\alpha$. We can now realise the SO(3,2) algebra by writing

$$\begin{aligned} M_{ab} &= -\frac{1}{8}(\sigma_{ab})^{\alpha\beta}y_\alpha y_\beta - \frac{1}{8}(\sigma_{ab})^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} \\ P_a &= \frac{1}{4}(\sigma_a)^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}\,, \end{aligned} \tag{2.12}$$

and using the $\star$ product in the commutators. As a result the SO(3,2) connection E defined in (2.2) reads

$$E = \frac{i}{2}\left[e^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}} + \frac{1}{2}\left(\omega^{\alpha\beta}y_\alpha y_\beta + \omega^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}}\right)\right], \tag{2.13}$$

where $e^{\alpha\dot{\alpha}}$ and $\omega^{\alpha\beta}$ are related to e^a and ω^{ab} via the relations (A.11) and (A.12). Useful relations are

$$e_\mu = -\frac{i}{4}(\sigma_\mu)^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}, \qquad e_\mu{}^{\alpha\dot{\alpha}} = -\frac{1}{2}(\sigma_\mu)^{\alpha\dot{\alpha}}\,. \tag{2.14}$$

The set of arbitrary polynomials $U(y,\bar{y})$ which obey projection and reality conditions according to

$$\tau\{U(y,\bar{y})\} \equiv U(iy,i\bar{y}) = -U(y,\bar{y}), \qquad U(y,\bar{y})^\dagger = -U(y,\bar{y})\,, \tag{2.15}$$

form a Lie algebra with respect to the commutator $[U,V]_\star$, denoted by $hs(4)$ in [10]. The algebra closes since $\tau(U\star V) = \tau(V)\star\tau(U)$, which follows from (2.11), and $(U\star V)^\dagger = V^\dagger\star U^\dagger$. The τ-projection restricts the polynomials to sums of monomials of degree $2+4\ell$, $\ell = 0,1,2\ldots$, containing generators with spin $s = 1+2\ell$. In order to gauge $hs(4)$, one introduces an $hs(4)$-valued connection $A = A_\mu(x;y,\bar{y})dx^\mu$ defined by

$$\begin{aligned} A_\mu(x;y,\bar{y}) &= \frac{i}{2}\sum_{\substack{m,n,\ell\geq 0\\ m+n=2+4\ell}}\frac{1}{m!n!}A_{\mu\,\alpha_1\ldots\alpha_m\dot{\alpha}_1\ldots\dot{\alpha}_n}(x)y^{\alpha_1}\ldots y^{\alpha_m}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_n}. && (2.16) \\ &= e+\omega+W\,, && (2.17) \end{aligned}$$

where W contains the higher spin gauge fields. It is convenient to define the SO(3,2) covariant field strength

$$\mathcal{F} \equiv dW + \{E,W\}_\star = \nabla W + \{e,W\}_\star\,. \tag{2.18}$$

The higher spin algebra $hs(4)$ has a unitary irreducible representation containing massless fields in AdS_4 with spins $s = 0,2,4,\ldots$, of which the spin $s\geq 2$ sector is realised as the physical degrees of freedom in A. In order to accommodate the physical scalar field one introduces a 0-form $\Phi(x;y,\bar{y})$ in the quasi-adjoint representation, defined by

$$\tau\{\Phi(y,\bar{y}\} = \bar{\pi}\{\Phi(y,\bar{y})\} \equiv \Phi(y,-\bar{y})\,, \qquad \Phi(y,\bar{y})^\dagger = \pi\{\Phi(y,\bar{y})\} \equiv \Phi(-y,\bar{y})\,, \tag{2.19}$$

and so has the structure

$$\Phi(x;y,\bar{y}) = \sum_{\substack{\ell+1,m,n\geq 0\\ |m-n|=4(\ell+1)}} \frac{1}{m!n!}\Phi_{\alpha_1\ldots\alpha_m\dot{\alpha}_1\ldots\dot{\alpha}_n}(x)y^{\alpha_1}\ldots y^{\alpha_m}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_n}\,. \tag{2.20}$$

The $\mathrm{SO}(3,2)$ covariant derivative of Φ is defined as

$$\mathcal{D}\Phi \equiv d\Phi + E\star\Phi - \Phi\star\pi\{E\} = \nabla\Phi + \{e,\Phi\}_\star\,. \tag{2.21}$$

As we will see below, the level $\ell = 0$ components are the Weyl tensor and its derivatives, which we introduced in the previous section, and at level $\ell > 0$ reside higher spin generalisations thereof. The physical scalar and its derivatives are contained at level $\ell = -1$.

Assuming that Φ and W are weak fields, the constraints on A and Φ leading to spacetime dynamics have the following expansion [9, 10, 19]

$$\begin{aligned}
\mathcal{R}_{\mu\nu} + \mathcal{F}_{\mu\nu} &= -2W_{[\mu}\star W_{\nu]} - i[R_{\mu\nu}{}^{\alpha\beta}\widehat{A}_\alpha\star\widehat{A}_\beta + \text{h.c.}]_{z=0} - \\
&\quad -2\sum_{n=1}^{\infty}\sum_{j=0}^{n}\Big((\widehat{e}+\widehat{W})^{(j)}_{[\mu}\star(\widehat{e}+\widehat{W})^{(n-j)}_{\nu]}\Big)_{z=0}\,, \qquad (2.22)\\
\nabla_\mu\Phi + \{e_\mu,\Phi\}_\star &= \Phi\star\bar{\pi}\{W_\mu\} - W_\mu\star\Phi + \\
&\quad +\sum_{n=2}^{\infty}\sum_{j=1}^{n}\Big(\widehat{\Phi}^{(j)}\star\bar{\pi}\{(\widehat{e}+\widehat{W})^{(n-j)}_\mu\} - (\widehat{e}+\widehat{W})^{(n-j)}_\mu\star\widehat{\Phi}^{(j)}\Big)_{z=0}\,. \qquad (2.23)
\end{aligned}$$

Here hatted quantities depend on $y,\bar{y}$ as well as an auxiliary set of oscillators $z_\alpha, \bar{z}_{\dot{\alpha}}$ obeying the following algebra

$$\widehat{f}\star\widehat{g} = \widehat{f}e^{i(\overleftarrow{\partial}^\alpha_+\overrightarrow{\partial}^-_\alpha + \overleftarrow{\partial}^{\dot{\alpha}}_-\overrightarrow{\partial}^+_{\dot{\alpha}})}\widehat{g}\,, \tag{2.24}$$

where $\partial_\pm = \partial^\pm = \partial_z \pm \partial_y$ and $\widehat{f} = \widehat{f}(y,\bar{y},z,\bar{z})$ and $\widehat{g} = \widehat{g}(y,\bar{y},z,\bar{z})$ are Weyl ordered functions (see appendix B), and the expansions $\widehat{e}_\mu = \sum_{j=0}^\infty \widehat{e}^{(j)}_\mu$ and $\widehat{W}_\mu = \sum_{j=0}^\infty \widehat{W}^{(j)}_\mu$ are given by

$$\widehat{e}_\mu = \frac{1}{1+\widehat{L}^{(1)}+\widehat{L}^{(2)}+\ldots}e_\mu\,, \qquad \text{and} \qquad \widehat{W}_\mu = \frac{1}{1+\widehat{L}^{(1)}+\widehat{L}^{(2)}+\ldots}W_\mu\,, \tag{2.25}$$

where

$$\widehat{L}^{(n)}(\widehat{f}) = i\int_0^1\frac{dt}{t}\Big(\widehat{A}^{(n)}_\alpha\star\partial^\alpha_-\widehat{f} + \partial^\alpha_+\widehat{f}\star\widehat{A}^{(n)}_\alpha + \widehat{A}^{(n)}_{\dot{\alpha}}\star\partial^{\dot{\alpha}}_+\widehat{f} + \partial^{\dot{\alpha}}_-\widehat{f}\star\widehat{A}^{(n)}_{\dot{\alpha}}\Big)_{z\to tz}\,. \tag{2.26}$$

The quantities $\widehat{A}^{(n)}_\alpha$ and $\widehat{\Phi}^{(n)}$ are given by

$$\begin{aligned}
\widehat{A}^{(0)}_\alpha &= 0\\
\widehat{\Phi}^{(1)} &= \Phi(y,\bar{y})\\
\widehat{A}^{(1)}_\alpha &= -\frac{ib_1}{2}z_\alpha\int_0^1 tdt\Phi(-tz,\bar{y})\kappa(tz,y)
\end{aligned}$$

$$\widehat{\Phi}^{(n)} = z^\alpha \sum_{j=1}^{n-1} \int_0^1 dt \Big(\widehat{\Phi}^{(j)} \star \bar{\pi}\{\widehat{A}_\alpha^{(n-j)}\} - \widehat{A}_\alpha^{(n-j)} \star \widehat{\Phi}^{(j)} \Big)_{z\to tz} +$$

$$+ \bar{z}^{\dot{\alpha}} \sum_{j=1}^{n-1} \int_0^1 dt \Big(\widehat{\Phi}^{(j)} \star \pi\{\widehat{A}_{\dot{\alpha}}^{(n-j)}\} - \widehat{A}_{\dot{\alpha}}^{(n-j)} \star \widehat{\Phi}^{(j)} \Big)_{z\to tz}$$

$$\widehat{A}_\alpha^{(n)} = z_\alpha \int_0^1 tdt \Big(-\frac{i}{2}\mathcal{V}^{(n)}(\widehat{\Phi} \star \kappa) + \sum_{j=1}^{n-1} \widehat{A}^{(j)\beta} \star \widehat{A}_\beta^{(n-j)} \Big)_{z\to tz} +$$

$$+ \bar{z}^{\dot{\beta}} \sum_{j=1}^{n-1} \int_0^1 tdt \Big[\widehat{A}_\alpha^{(j)}, \widehat{A}_{\dot{\beta}}^{(n-j)} \Big]_{\star, z\to tz} \tag{2.27}$$

and $\widehat{A}_{\dot{\alpha}}^{(n)} = -(\widehat{A}_\alpha^{(n)})^\dagger$. The function $\mathcal{V}(\widehat{\Phi} \star \kappa)$, where $\kappa(y,z) = \exp(iy^\alpha z_\alpha)$, has to be odd. Already the simplest choice of $\mathcal{V}$, namely a linear function, leads to highly nontrivial interactions in the right hand sides of (2.22) and (2.23). Adding a $(2n+1)^{\rm th}$ order term to $\mathcal{V}$ leads to modifications of the interactions starting at the $(2n+1)^{\rm th}$ order. Whether these are genuine interaction ambiguities, or can be removed by field redefinitions is not known. In the former case we expect the ambiguity to be determined once the theory is compared with its holographic dual, or some more fundamental formulation of the theory in the bulk. Since we will focus on quadratic contributions in the scalar field we can assume that $\mathcal{V}(\widehat{\Phi} \star \kappa) = b_1 \widehat{\Phi} \star \kappa$, where b_1 is the first order expansion coefficient.

The constraints (2.22) and (2.23) follow from solving an extended set of constraints on $\widehat{\Phi}$ and

$$\widehat{A} = (\widehat{A}_\mu + [i\omega_\mu^{\alpha\beta} \widehat{A}_\alpha \star \widehat{A}_\beta - \text{h.c.}])dx^\mu + \widehat{A}_\alpha dz^\alpha + \widehat{A}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} , \tag{2.28}$$

which are forms living on spacetime times an internal manifold for which $z, \bar{z}$ are coordinates. Besides being consistent with the τ and reality conditions, the basic property of (2.22) and (2.23) is that they are integrable order by order in the weak field expansion. Note that this ensures invariance under higher spin gauge transformations and diffeomorphisms (which are incorporated into the gauge group as field dependent gauge transformations). The rationale behind the expansion of the μ component in (2.28) is that it implies that the constraints are manifestly invariant under local Lorentz transformations under which the component fields in e and W transform as Lorentz tensors and ω as the Lorentz connection [19, 10].

The linearised form of (2.22) contains the physical field equations for spin $s = 2, 4, \ldots$ and algebraic equations for the auxiliary gauge fields [9] and $\Phi_{\alpha_1\ldots\alpha_{2s}}$, which are the Weyl tensor and its higher spin generalisations. The linearised form of (2.23) reads $\nabla_\mu \Phi + \{e_\mu, \Phi\} = 0$ where e_μ is given by (2.14), which can be written in components as

$$\nabla_\mu \Phi_{\alpha_1\ldots\alpha_m\dot{\alpha}_1\ldots\dot{\alpha}_n} = \frac{i}{2} mn (\sigma_\mu)_{\alpha_1\dot{\alpha}_1} \Phi_{\alpha_2\ldots\alpha_m\dot{\alpha}_2\ldots\dot{\alpha}_n} - \frac{i}{2}(\sigma_\mu)^{\beta\dot{\beta}} \Phi_{\beta\alpha_1\ldots\alpha_m\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_n} , \tag{2.29}$$

where separate symmetrisation of the dotted and undotted indices on the right hand side is assumed. From (2.29) it follows that $\Phi_{\alpha_1\ldots\alpha_{2s+k}\dot{\alpha}_1\ldots\dot{\alpha}_k}$, $s = 0, 2, 4, \ldots$, can be expressed in terms of k derivatives of $\Phi_{\alpha_1\ldots\alpha_{2s}}$. As an example, we see that $\nabla \Phi_{\alpha_1\ldots\alpha_4} = 2ie^{\beta\dot{\beta}} \Phi_{\alpha_1\ldots\alpha_4\beta\dot{\beta}}$, which is nothing but equation (2.8). For $s = 0$ one finds

$$\Phi_{\alpha_1\ldots\alpha_k\dot\alpha_1\ldots\dot\alpha_k} = (-i)^k(\sigma^{\mu_1})_{\alpha_1\dot\alpha_1}\ldots(\sigma^{\mu_k})_{\alpha_k\dot\alpha_k}\nabla_{\mu\{k\}}\phi\,, \tag{2.30}$$

where we use the notation defined in (1.2). Note that the dotted and undotted indices on the right hand side are automatically symmetrised since the Lorentz vector indices are traceless and symmetric. From the linearised form of (2.23) it also follows that ϕ is the physical scalar with the linearised field equation

$$\nabla^\mu\nabla_\mu\phi = -2\phi\,. \tag{2.31}$$

The mass $m^2 = -2$ corresponds to a scalar lowest weight state with AdS energy $E = 1$.

By working out the $\star$ products in the above relations, and solving for the auxiliary fields order by order in the weak field expansion, it is possible to extract the field equations describing the full interacting massless higher spin theory to any desired order by means of straightforward albeit increasingly tedious calculations.

3. Scalar field terms in the Einstein equation

We will now consider the expansion scheme to second order and calculate all quadratic contributions to the Einstein equation from the scalar field ϕ and its derivatives.

One can show that upon linearising the right hand side of (2.22) in the weak fields one obtains (2.5) and (2.6) provided that $b_1 = 1$. For a general complex b_1 the right hand side of the $\mathcal{R}^{ab}$ constraint is modified though its structure remains the same. To the second order, (2.22) implies that

$$\mathcal{R}^{ab}(\omega,e) = \frac{1}{2}(\sigma^{ab})^{\alpha\beta}J_{\alpha\beta} - \text{h.c.} \tag{3.1}$$

$$\mathcal{R}^{a}(\omega,e) = (\sigma^{a})^{\alpha\dot\alpha}J_{\alpha\dot\alpha}\,, \tag{3.2}$$

where

$$J_{\alpha\beta} = \frac{\partial^2}{\partial y^\alpha\partial y^\beta}\,J|_{y=0} \qquad\text{and}\qquad J_{\alpha\dot\alpha} = \frac{\partial^2}{\partial y^\alpha\partial \bar y^{\dot\alpha}}\,J|_{y=0}\,, \tag{3.3}$$

and the 2-form $J = J_{\mu\nu}dx^\mu dx^\nu$ is given by

$$J_{\mu\nu} = 2i\{e_{[\mu}{}^{\alpha\dot\alpha}e_{\nu]}{}^{\beta}{}_{\dot\alpha}\widehat{A}^{(1)}_\alpha\star\widehat{A}^{(1)}_\beta + \text{h.c.}\}_{z=0} + L_{\mu\nu} + W\text{-terms}\,, \tag{3.4}$$

where

$$L_{\mu\nu} = -\left(\left[\widehat{L}^{(1)}(e_\mu),\widehat{L}^{(1)}(e_\nu)\right]_\star - 2\left[e_{[\mu},\widehat{L}^{(2)}(e_{\nu]})\right]_\star + 2\left[e_{[\mu},\widehat{L}^{(1)}\circ\widehat{L}^{(1)}(e_{\nu]})\right]_\star\right)_{z=0}. \tag{3.5}$$

In (3.4) we have used the background value of $R_{\mu\nu}$ which is given by $R_{\mu\nu}{}^{\alpha\beta} = -2e_{[\mu}{}^{\alpha\dot\alpha}e_{\nu]}{}^{\beta}{}_{\dot\alpha}$, as follows from (2.4). Writing $\omega^{ab} = \omega^{ab}(e)+\kappa^{ab}$, where $\omega^{ab}(e)$ is the Levi-Civita connection obeying $\mathcal{R}^a(\omega(e),e) = 0$, and κ^{ab} is contorsion, we have

$$\mathcal{R}^{ab}(\omega,e) = \mathcal{R}^{ab}(\omega(e),e) - i\nabla\kappa^{ab}$$

$$\mathcal{R}^{a}(\omega,e) = -ie^b\wedge\kappa_b{}^a\,. \tag{3.6}$$

By taking the trace of (3.1) one obtains a field equation containing the Ricci tensor,

$$\begin{aligned}\mathrm{Ric}_{\mu\nu}+3g_{\mu\nu} &= 2\nabla_{[\rho}\kappa_{\mu]\nu}{}^{\rho}-\frac{1}{2}\Big\{(\sigma_\mu)^{\gamma\dot\alpha}(\sigma_\nu)^{\delta\dot\beta}\Big(\widehat{A}^{(1)}_{\dot\alpha}\star\widehat{A}^{(1)}_{\dot\beta}\Big)_{\gamma\delta}+\text{h.c.}\Big\}_{z=0}+\\ &\quad+\frac{i}{2}\Big\{(\sigma_\mu{}^{\rho})_{\alpha\beta}L_{\nu\rho}{}^{\alpha\beta}-\text{h.c.}\Big\},\end{aligned}\tag{3.7}$$

where symmetrisation of μ and ν is understood in each term, and we have used (2.14). To obtain (3.7) we picked the symmetric part of the trace of eq. (3.1). The antisymmetric part should be identically satisfied, for the system not to be overdetermined. We have not attempted to prove the identity. Upon substituting (3.6) in (3.2) and solving for the contorsion, we find

$$\kappa_\nu{}^{ab}=\frac{i}{2}\Big((\sigma_\nu)^{\alpha\dot\beta}J^{ab}{}_{\alpha\dot\beta}+(\sigma^a)^{\alpha\dot\beta}J_\nu{}^b{}_{\alpha\dot\beta}-(\sigma^b)^{\alpha\dot\beta}J_\nu{}^a{}_{\alpha\dot\beta}\Big)\,.\tag{3.8}$$

In order to obtain the contributions to the stress-energy tensor that are quadratic in the scalar we henceforth drop the contributions to J from W and all components in Φ which have different number of dotted and undotted indices. The remaining scalar contributions to the stress-energy tensor is a sum of terms containing the structures $\Phi_{\alpha_1\ldots\alpha_k\dot\alpha_1\ldots\dot\alpha_k}\Phi_{\alpha_1\ldots\alpha_\ell\dot\alpha_1\ldots\dot\alpha_\ell}$, where $\Phi_{\alpha_1\ldots\alpha_k\dot\alpha_1\ldots\dot\alpha_k}$ can be substituted using (2.30). To compute the various terms in the quantity $L_{\mu\nu}$ given in (3.5) we make use of (2.24)–(2.27). We first obtain

$$\widehat{L}^{(1)}(e_\mu)=-\frac{b_1}{2}z^\alpha e_{\mu,\alpha\dot\alpha}\bar\partial^{\dot\alpha}_y\int_0^1\!\!\int_0^1 dt'tdt\Phi(-tt'z,\bar y)\kappa(tt'z,y)-\text{h.c.}\tag{3.9}$$

At this stage it is convenient to introduce

$$\widehat{\mathcal{L}}^{(n)}(\widehat{f})\equiv -i\int_0^1\frac{dt}{t}\Big(\widehat{A}^{(n)}_\alpha\star\partial^\alpha_-\widehat{f}-\partial^\alpha_+\widehat{f}\star\widehat{A}^{(n)}_\alpha\Big)_{z\to tz}\tag{3.10}$$

such that we can write,

$$\widehat{L}^{(n)}=\widehat{\mathcal{L}}^{(n)}+\widehat{\mathcal{L}}^{\dagger(n)}\,.\tag{3.11}$$

We then find

$$\Big[\widehat{L}^{(1)}(e_\mu),\widehat{L}^{(1)}(e_\nu)\Big]_{\star,z=0}=\Big[\widehat{\mathcal{L}}^{(1)}(e_\mu),\widehat{\mathcal{L}}^{(1)}(e_\nu)\Big]_{\star,z=0}+\Big[\widehat{\mathcal{L}}^{\dagger(1)}(e_\mu),\widehat{\mathcal{L}}^{\dagger(1)}(e_\nu)\Big]_{\star,z=0},\tag{3.12}$$

modulo the omitted terms, as explained above. Furthermore, we have

$$\Big[e_\mu,\widehat{L}^{(2)}(e_\nu)\Big]_{\star,z=0}=\Big[e_\mu,\widehat{\mathcal{L}}^{(2)}(e_\nu)\Big]_{\star,z=0}+\Big[e_\mu,\widehat{\mathcal{L}}^{\dagger(2)}(e_\nu)\Big]_{\star,z=0},\tag{3.13}$$

where

$$\Big[e_\mu,\widehat{\mathcal{L}}^{(2)}(e_\nu)\Big]_{\star,z=0}=-\frac{1}{2}e_\mu{}^{\beta\dot\beta}e_\nu{}^{\alpha\dot\alpha}\Big[y_\beta\bar y_{\dot\beta},\int_0^1\frac{dt}{t}\bar\partial^y_{\dot\alpha}\widehat{A}^{(2)}_\alpha(z\to tz)\Big]_{\star,z=0}\tag{3.14}$$

and

$$\widehat{A}^{(2)}_\alpha=\frac{z_\alpha}{2}\int_0^1 tdt\Big(\Big[\widehat{A}^{(1)\delta},\widehat{A}^{(1)}_\delta\Big]_\star-ib_1\widehat{\Phi}^{(2)}\star\kappa\Big)(z\to tz)+\widehat{B}_\alpha\,,\tag{3.15}$$

denoting by $\widehat{B}_\alpha$ terms in $\widehat{A}^{(2)}_\alpha$ that do not contribute once z is set to zero. Finally, considering

$$\widehat{L}^{(1)}\circ\widehat{L}^{(1)}(e_\mu)=-\int_0^1\frac{dt}{t}\left(\left\{\widehat{A}^{(1)\delta},\partial^{(z)}_\delta\widehat{L}^{(1)}(e_\mu)\right\}_\star-\left[\widehat{A}^{(1)\delta},\partial^{(y)}_\delta\widehat{L}^{(1)}(e_\mu)\right]_\star\right)(z\to tz)+\text{h.c.},\tag{3.16}$$

we find that

$$\left[e_\mu,\widehat{L}^{(1)}\circ\widehat{L}^{(1)}(e_\nu)\right]_{\star,z=0}=\left[e_\mu,\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]_{\star,z=0}+\left[e_\mu,\widehat{\mathcal{L}}^{\dagger(1)}\circ\widehat{\mathcal{L}}^{\dagger(1)}(e_\nu)\right]_{\star,z=0}.\tag{3.17}$$

From the above analysis we conclude that

$$J_{\mu\nu}=\mathcal{J}_{\mu\nu}-\mathcal{J}^\dagger_{\mu\nu}\,,\tag{3.18}$$

where

$$\mathcal{J}_{\mu\nu}=2i\{e_{[\mu}{}^{\alpha\dot\alpha}e_{\nu]}{}^\beta{}_{\dot\alpha}\widehat{A}^{(1)}_\alpha\star\widehat{A}^{(1)}_\beta\}_{z=0}+\mathcal{L}_{\mu\nu}\tag{3.19}$$

and

$$\mathcal{L}_{\mu\nu}=-\left(\left[\widehat{\mathcal{L}}^{(1)}(e_\mu),\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]_\star-2\left[e_{[\mu},\widehat{\mathcal{L}}^{(2)}(e_{\nu]})\right]_\star+2\left[e_{[\mu},\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_{\nu]})\right]_\star\right)_{z=0}.\tag{3.20}$$

The explicit calculation of (3.12), (3.14) and (3.17) is straightforward but lengthy. The details are given in appendix C. The result is a sum of various contractions of $\Phi_{\alpha_1\ldots\alpha_k\dot\alpha_1\ldots\dot\alpha_k}$ $\Phi_{\alpha_1\ldots\alpha_\ell\dot\alpha_1\ldots\dot\alpha_\ell}$ for $|k-\ell|=0$ or 2. Finally, after converting spinor indices to vector indices, as described in appendix D, we arrive at

$$\begin{aligned}\text{Ric}_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-3g_{\mu\nu}\;=\;\text{Re}\{b_1^2\}\Bigg[\sum_k\frac{2^k}{(k!)^2}\Bigg(&\xi(k)\,g_{\mu\nu}\nabla_{\rho\{k+1\}}\,\phi\;\nabla^{\rho\{k+1\}}\,\phi\,+\\&+\eta(k)\nabla_{\rho\{k\}\mu}\,\phi\;\nabla^{\rho\{k\}}{}_\nu\,\phi\,+\\&+\zeta(k)\nabla_{\rho\{k\}\mu\nu}\phi\nabla^{\rho\{k\}}\,\phi\Bigg)-\frac{4}{9}\;g_{\mu\nu}\phi\,\phi\Bigg]\,,\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}\xi(k)\;&=\;-\frac{1}{24}\{132k^{10}+4169k^9+57902k^8+464477k^7+2378336k^6+8109935k^5+\\&\quad+18627566k^4+28429503k^3+27570000k^2+15326604k+\\&\quad+3703824\}\Big/\{(k+5)^2(k+4)^2(k+3)^2(k+2)^2(k+1)^2\},\end{aligned}\tag{3.22}$$

$$\begin{aligned}\eta(k)\;&=\;\frac{1}{3}\{42k^9+1234k^8+15738k^7+114011k^6+515273k^5+1500759k^4+\\&\quad+2804017k^3+3224520k^2+2060706k+554772\}\Big/\\&\quad\{(k+5)^2(k+4)^2(k+3)^2(k+2)^2(k+1)\}\,,\end{aligned}\tag{3.23}$$

$$\begin{aligned}\zeta(k)\;&=\;\frac{1}{6}\{12k^8+379k^7+5047k^6+36860k^5+161255k^4+433379k^3+701764k^2+\\&\quad+629748k+240912\}\Big/\{(k+5)^2(k+4)^2(k+3)^2(k+2)^2\}\,.\end{aligned}\tag{3.24}$$

We note that for $k \gg 1$

$$\begin{aligned} \xi(k) &\sim -\frac{11}{2} \\ \eta(k) &\sim 14 \\ \zeta(k) &\sim 2\,. \end{aligned} \tag{3.25}$$

4. Summary and discussion

We have calculated the scalar field content of the Einstein equation in AdS_4 higher spin gauge theory. This constitutes a first step towards finding the graviton-ϕ^2 terms in the action. The details of the calculation are given in the appendices, since we believe they can be useful in making further calculations, perhaps implemented on a computer.

In order to compute the cubic action in the spin $s = 0, 2$ sector we would also have to calculate graviton-scalar terms in the scalar field equation. Then these two contributions in the field equations should have identical coefficients since they originate from the same graviton-ϕ^2 terms in the action. It is also possible that some recombination of the spin $s = 0, 2$ field equations is required in order to satisfy the above integrability conditions for the action. The existence of an action is not necessarily contradicted by the higher derivative terms in (3.21), since these terms may arise from a term of the form $G^{\mu\nu}(g)\partial_\mu\phi\partial_\nu\phi$, where $G_{\mu\nu}(g)$ depends on higher derivatives of the metric while $G_{\mu\nu}(g)$ and $\nabla^\mu G_{\mu\nu}(g)$ vanishes in the AdS background.

Starting from a general $\mathcal{V}(X)$ we have found that the contribution to the stress-energy tensor which is quadratic in the scalar field is proportional to $\mathrm{Re}\{b_1^2\}$. This raises the issue of the positivity of the Killing energy associated with the stress-energy tensor, since it is in fact possible for $\mathrm{Re}\{b_1^2\}$ to be either positive or negative depending on the parity of the scalar field [20]. Moreover, the analysis of the Killing energy functional might depend on the choice of the boundary condition for the scalar field. We defer these issues to a future publication.

It would be interesting to consider the influence of this scalar field in a cosmological context. Close to the Big Bang more symmetries were realised, and it might be possible that a higher spin theory is needed to understand the dynamics. In this point of view the scalar field ϕ investigated here may have connections to the inflation. For this reason and also for understanding better the bulk description of the $\mathrm{O}(N)$ model RG-flows, it would be interesting to find domain wall solutions to the full field equations in which all fields are depending on one space or time coordinate.

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A. Conventions and useful relations

A.1 Spinor conventions

We always use symmetrisations and anti-symmetrisations with unit strength. We define the SL$(2,C)$ invariant $\epsilon_{\alpha\beta}$ by

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = (\epsilon_{\dot\alpha\dot\beta})^\dagger , \qquad \epsilon_{\alpha\beta}\epsilon^{\delta\gamma} = 2\delta^{\delta\gamma}_{\alpha\beta} . \tag{A.1}$$

Spinor index contraction is according to the north-west south-east rule. In particular,

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta , \qquad \text{and} \qquad \psi_\alpha = \psi^\beta\epsilon_{\beta\alpha} . \tag{A.2}$$

The van der Waerden symbols $(\sigma^\mu)_{\alpha\dot\beta}$ are defined as

$$(\sigma^{(\mu})_\alpha{}^{\dot\alpha}(\sigma^{\nu)})_{\beta\dot\alpha} = \eta^{\mu\nu}\epsilon_{\alpha\beta} , \qquad \Big((\sigma^\mu)_{\alpha\dot\beta}\Big)^\dagger = (\sigma^\mu)_{\beta\dot\alpha} . \tag{A.3}$$

We also define the following matrices

$$\begin{aligned}
(\sigma^{\mu\nu})_{\alpha\beta} &= (\sigma^{[\mu})_\alpha{}^{\dot\alpha}(\sigma^{\nu]})_{\beta\dot\alpha} , && \text{(A.4)}\\
(\sigma^{\mu\nu\rho})_{\alpha\dot\beta} &= (\sigma^{[\mu})_\alpha{}^{\dot\alpha}(\sigma^\nu)^\beta{}_{\dot\alpha}(\sigma^{\rho]})_{\beta\dot\beta} , && \text{(A.5)}\\
(\sigma^{\mu\nu\rho\tau})_{\alpha\beta} &= (\sigma^{[\mu})_\alpha{}^{\dot\alpha}(\sigma^\nu)^\gamma{}_{\dot\alpha}(\sigma^\rho)_\gamma{}^{\dot\gamma}(\sigma^{\tau]})_{\beta\dot\gamma} , && \text{(A.6)}\\
(\sigma^{\mu\nu})_{\dot\alpha\dot\beta} &= \Big((\sigma^{\mu\nu})_{\alpha\beta}\Big)^\dagger , && \text{(A.7)}\\
(\sigma^{\mu\nu\rho\tau})_{\dot\alpha\dot\beta} &= \Big((\sigma^{\mu\nu\rho\tau})_{\alpha\beta}\Big)^\dagger . && \text{(A.8)}
\end{aligned}$$

One can show that

$$\begin{aligned}
\Big((\sigma^{\mu\nu\rho})_{\alpha\dot\beta}\Big)^\dagger &= -(\sigma^{\mu\nu\rho})_{\beta\dot\alpha} ,\\
(\sigma^{\mu\nu\rho\tau})_{\alpha\beta} &= i\epsilon^{\mu\nu\rho\tau}\epsilon_{\alpha\beta} .
\end{aligned} \tag{A.9}$$

From the defining relation (A.3) it follows that

$$\begin{aligned}
\sigma^{\mu_1\dots\mu_m}\sigma_{\nu_1\dots\nu_n} &= \sigma^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} + mn\,\delta^{[\mu_m}_{[\nu_1}\sigma^{\mu_1\dots\mu_{m-1}]}{}_{\nu_2\dots\nu_n]} + \dots +\\
&\quad + \binom{m}{k}\binom{n}{k}k!\;\delta^{[\mu_m\dots\mu_{m-k+1}}_{[\nu_1\dots\nu_k}\sigma^{\mu_1\dots\mu_{m-k}]}{}_{\nu_{k+1}\dots\nu_n]} +\\
&\quad + \;\cdots .
\end{aligned} \tag{A.10}$$

Vectors and antisymmetric tensors are expressed in spinor indices according to

$$V^\mu = (\sigma^\mu)_{\alpha\dot\alpha}V^{\alpha\dot\alpha} , \qquad V^{\alpha\dot\alpha} = -\frac{1}{2}(\sigma_\mu)^{\alpha\dot\alpha}V^\mu , \tag{A.11}$$

$$A^{\mu\nu} = \frac{1}{2}\Big((\sigma^{\mu\nu})_{\alpha\beta}A^{\alpha\beta} + (\sigma^{\mu\nu})_{\dot\alpha\dot\beta}A^{\dot\alpha\dot\beta}\Big) , \qquad A^{\alpha\beta} = \frac{1}{4}(\sigma_{\mu\nu})^{\alpha\beta}A^{\mu\nu} . \tag{A.12}$$

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A.2 Notation for symmetrised spinor indices

In the following we shall use a condensed notation for symmetrisation of spinor indices, defined by

$$f_{\alpha(m)} = f_{\alpha_1\ldots\alpha_m} = \frac{1}{m!}\sum_P f_{\alpha_{P(1)}\ldots\alpha_{P(m)}}\,, \tag{A.13}$$

and

$$f_{\alpha(m_1)}g_{\alpha(m_2)} = \frac{1}{(m_1+m_2)!}\sum_P f_{\alpha_{P(1)}\cdots\alpha_{P(m_1)}}g_{\alpha_{P(m_1+1)}\cdots\alpha_{P(m_1+m_2)}}\,, \tag{A.14}$$

where the right hand sides are summed over all permutations P of indices. Note that within this notation there is no symmetrisation in $f_{\alpha_1(m_1)}g_{\alpha_2(m_2)}$ between the indices of type α_1 and those of type α_2.

A.3 Useful relations

From (A.10) we can derive the following relations.

$$\begin{aligned}
(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)_{\beta\dot\beta} &= -\frac{1}{2}\Big(\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + (\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \\
&\qquad + \ (\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta} + (\sigma^{\mu\rho})_{\alpha\beta}(\sigma_\rho{}^\nu)_{\dot\alpha\dot\beta}\Big) \\
(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)_{\beta\dot\beta} &= -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \\
(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})^{\delta\gamma} &= 8\epsilon_\alpha{}^{(\delta}\epsilon_\beta{}^{\gamma)} \\
(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})^{\dot\delta\dot\gamma} &= 0 \\
(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_\mu)^{\delta\dot\delta}(\sigma_\nu)^{\gamma\dot\gamma} &= -4\epsilon^{\dot\delta\dot\gamma}\epsilon_\alpha{}^{(\delta}\epsilon_\beta{}^{\gamma)} \\
(\sigma^{\mu\nu})_{\alpha\beta}(\sigma_\mu{}^\rho)^{\gamma\delta} &= 2\epsilon_{(\alpha}{}^{(\gamma}(\sigma^{\nu\rho})_{\beta)}{}^{\delta)}\,.
\end{aligned} \tag{A.15}$$

B. Evaluations of $\star$ -products

We begin by observing the following useful formulae

$$\begin{aligned}
\partial_\alpha y_\beta &= \epsilon_{\alpha\beta} & \qquad \partial^\alpha y^\beta &= \epsilon^{\alpha\beta} \\
\partial_\alpha y^\beta &= \delta_\alpha^\beta & \qquad \partial^\alpha y_\beta &= -\delta_\beta^\alpha.
\end{aligned} \tag{B.1}$$

The star product (2.24) is equivalent to the following contraction rules between the y and z oscillators

$$\begin{aligned}
y_\alpha\star y_\beta &= y_\alpha y_\beta + i\epsilon_{\alpha\beta} & \qquad y_\alpha\star z_\beta &= y_\alpha z_\beta - i\epsilon_{\alpha\beta} \\
z_\alpha\star y_\beta &= z_\alpha y_\beta + i\epsilon_{\alpha\beta} & \qquad z_\alpha\star z_\beta &= z_\alpha z_\beta - i\epsilon_{\alpha\beta} \\
\\
\bar y_{\dot\alpha}\star\bar y_{\dot\beta} &= \bar y_{\dot\alpha}\bar y_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta} & \qquad \bar z_{\dot\alpha}\star\bar y_{\dot\beta} &= \bar z_{\dot\alpha}\bar y_{\dot\beta} - i\epsilon_{\dot\alpha}\dot\beta \\
\bar y_{\dot\alpha}\star\bar z_{\dot\beta} &= \bar y_{\dot\alpha}\bar z_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta} & \qquad \bar z_{\dot\alpha}\star\bar z_{\dot\beta} &= \bar z_{\dot\alpha}\bar z_{\dot\beta} - i\epsilon_{\dot\alpha\dot\beta}\,.
\end{aligned} \tag{B.2}$$

Note that $[z_\alpha, y_\beta]_\star = 0$. The Weyl ordered product is denoted by

$$y_{\alpha(m)} = \frac{1}{m!}\sum_P y_{\alpha_{P(1)}}\star\cdots\star y_{\alpha_{P(m)}} = \underbrace{y_{\alpha(1)}\star\cdots\star y_{\alpha(1)}}_{m\text{ factors}}\,, \tag{B.3}$$

and equivalently for z. From the contraction rules above it follows that

$$y_{\alpha_1(m_1)} \star y_{\alpha_2(m_2)} = \sum_k i^k k! \begin{pmatrix} m_1 \\ k \end{pmatrix} \begin{pmatrix} m_2 \\ k \end{pmatrix} y_{\alpha_1(m_1-k)} y_{\alpha_2(m_2-k)} \epsilon_{\alpha_1(k)\alpha_2(k)} \,, \tag{B.4}$$

where

$$\epsilon_{\alpha(k)\beta(k)} = \frac{1}{k!} \sum_P \epsilon_{\alpha_1 \beta_{P(1)}} \cdots \epsilon_{\alpha_k \beta_{P(k)}} \,. \tag{B.5}$$

Indeed, from this we can verify that the $\star$ product is equivalent to the differential operator given in (2.11),

$$\star = 1 + i\epsilon_{\alpha\beta} \overleftarrow{\partial}^{\alpha} \overrightarrow{\partial}^{\beta} + \frac{i^2}{2!} \epsilon_{\alpha(2)\beta(2)} \overleftarrow{\partial}^{\alpha(2)} \overrightarrow{\partial}^{\beta(2)} + \cdots = \exp\{-i \overleftarrow{\partial}^{\alpha} \overrightarrow{\partial}_{\alpha}\} \,. \tag{B.6}$$

As an example, consider the product

$$\begin{aligned} y_{\alpha_1(2)} \star y_{\alpha_2(2)} &= \sum_k i^k k! \begin{pmatrix} 2 \\ k \end{pmatrix} \begin{pmatrix} 2 \\ k \end{pmatrix} y_{\alpha_1(2-k)} y_{\alpha_2(2-k)} \epsilon_{\alpha_1(k)\alpha_2(k)} \\ &= -2\epsilon_{\alpha_1\alpha_2(2)} + 4i y_{\alpha_1(1)} y_{\alpha_2(1)} \epsilon_{\alpha_1(1)\alpha_2(1)} + y_{\alpha_1(2)} y_{\alpha_2(2)} \\ &= -\left(\epsilon_{\alpha_{11}\alpha_{21}} \epsilon_{\alpha_{12}\alpha_{22}} + \epsilon_{\alpha_{11}\alpha_{22}} \epsilon_{\alpha_{12}\alpha_{21}}\right) + \\ &\quad + i(y_{\alpha_{11}} y_{\alpha_{22}} \epsilon_{\alpha_{12}\alpha_{21}} + y_{\alpha_{11}} y_{\alpha_{21}} \epsilon_{\alpha_{12}\alpha_{22}} + y_{\alpha_{12}} y_{\alpha_{22}} \epsilon_{\alpha_{11}\alpha_{21}} + y_{\alpha_{12}} y_{\alpha_{21}} \epsilon_{\alpha_{11}\alpha_{22}}) + \\ &\quad + y_{\alpha_{11}} y_{\alpha_{12}} y_{\alpha_{21}} y_{\alpha_{22}} \,. \end{aligned} \tag{B.7}$$

Expanding a Weyl ordered polynomial in y as

$$F(y) = \sum_m \frac{1}{m!} F_{\alpha(m)} y^{\alpha(m)} \,, \tag{B.8}$$

and using

$$\frac{\partial^{\alpha(m)}}{m!} y_{\beta(m_1)} y_{\gamma(m_2)}|_{y=0} = (-1)^m \delta^{\alpha(m)}_{\beta(m_1)\gamma(m_2)} \qquad \text{and} \qquad \partial^{\alpha(m)} F(y)|_{y=0} = F^{\alpha(m)} \,, \tag{B.9}$$

where $m_1 + m_2 = m$, we compute

$$\begin{aligned} F \star G &= \sum_{m_1,m_2} (-1)^{m_1+m_2} \frac{F^{\alpha_1(m_1)}}{m_1!} y_{\alpha_1(m_1)} \star \frac{G^{\alpha_2(m_2)}}{m_2!} y_{\alpha_2(m_2)} \\ &= \sum_{m_1,m_2} (-1)^{m_1+m_2} \frac{F^{\alpha_1(m_1)} G^{\alpha_2(m_2)}}{m_1! m_2!} \times \\ &\quad \times \sum_k i^k k! \begin{pmatrix} m_1 \\ k \end{pmatrix} \begin{pmatrix} m_2 \\ k \end{pmatrix} y_{\alpha_1(m_1-k)} y_{\alpha_2(m_2-k)} \epsilon_{\alpha_1(k)\alpha_2(k)} \,, \\ (F \star G)^{\alpha(m)} &= \sum_{m_1,m_2} \frac{F^{\alpha_1(m_1)} G^{\alpha_2(m_2)}}{m_1! m_2!} \sum_k i^k k! \begin{pmatrix} m_1 \\ k \end{pmatrix} \begin{pmatrix} m_2 \\ k \end{pmatrix} m! \delta^{\alpha(m)}_{\alpha_1(m_1-k)\alpha_2(m_2-k)} \epsilon_{\alpha_1(k)\alpha_2(k)} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{m_1,m_2,k} \frac{i^k m! k!}{(m_1+k)!(m_2+k)!} \binom{m_1+k}{k} \binom{m_2+k}{k} \times \\
&\quad \times F^{\alpha_1(m_1+k)} G^{\alpha_2(m_2+k)} \delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)} \epsilon_{\alpha_1(k)\alpha_2(k)} \\
&= \sum_{\substack{k \\ m_1+m_2=m}} \frac{i^k m!}{k! m_1! m_2!} F^{\alpha(m_1)}_{\gamma(k)} G^{\alpha(m_2)\gamma(k)} \,.
\end{aligned}
\tag{B.10}
$$

This is easily extended to general polynomials in $y, \bar{y}, z, \bar{z}$, which we expand as

$$
F(Y,Z) = \sum_{m,\tilde{m},n,\tilde{n}} \frac{F_{\alpha(m)\dot{\alpha}(\tilde{m}),\beta(n)\dot{\beta}(\tilde{n})}}{m!\tilde{m}!n!\tilde{n}!} y^{\alpha(m)} \bar{y}^{\dot{\alpha}(\tilde{m})} z^{\beta(n)} \bar{z}^{\dot{\beta}(\tilde{n})} \,,
\tag{B.11}
$$

where commas are used to separate the indices belonging to y's and z's. Using (B.2) we compute

$$
\begin{aligned}
(F \star G)^{\alpha(m)\dot{\alpha}(\tilde{m}),\beta(n)\dot{\beta}(\tilde{n})} &= \sum_{K(m,\tilde{m},n,\tilde{n})} C(K(m,\tilde{m},n,\tilde{n})) \times \\
&\quad \times F^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})\dot{\alpha}_1(\tilde{m}_1+\tilde{k}_{\alpha\alpha}+\tilde{k}_{\alpha\beta}),\beta_1(n_1+k_{\beta\alpha}+k_{\beta\beta})\dot{\beta}_1(\tilde{n}_1+\tilde{k}_{\beta\alpha}+\tilde{k}_{\beta\beta})} \times \\
&\quad \times G^{\alpha_2(m_2+k_{\alpha\alpha}+k_{\beta\alpha})\dot{\alpha}_2(\tilde{m}_2+\tilde{k}_{\alpha\alpha}+\tilde{k}_{\beta\alpha}),\beta_2(n_2+k_{\alpha\beta}+k_{\beta\beta})\dot{\beta}_2(\tilde{n}_2+\tilde{k}_{\alpha\beta}+\tilde{k}_{\beta\beta})} \times \\
&\quad \times \delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)} \delta^{\dot{\alpha}(\tilde{m})}_{\dot{\alpha}_1(\tilde{m}_1)\dot{\alpha}_2(\tilde{m}_2)} \delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)} \delta^{\dot{\beta}(\tilde{n})}_{\dot{\beta}_1(\tilde{n}_1)\dot{\beta}_2(\tilde{n}_2)} \\
&\quad \times \epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})} \epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})} \epsilon_{\beta_1(k_{\beta\alpha})\alpha_2(k_{\beta\alpha})} \epsilon_{\beta_1(k_{\beta\beta})\beta_2(k_{\beta\beta})} \times \\
&\quad \times \epsilon_{\dot{\alpha}_1(\tilde{k}_{\alpha\alpha})\dot{\alpha}_2(\tilde{k}_{\alpha\alpha})} \epsilon_{\dot{\alpha}_1(\tilde{k}_{\alpha\beta})\dot{\beta}_2(\tilde{k}_{\alpha\beta})} \epsilon_{\dot{\beta}_1(\tilde{k}_{\beta\alpha})\dot{\alpha}_2(\tilde{k}_{\beta\alpha})} \epsilon_{\dot{\beta}_1(\tilde{k}_{\beta\beta})\dot{\beta}_2(\tilde{k}_{\beta\beta})} \\
&= \sum_{K(m,\tilde{m},n,\tilde{n})} C(K(m,\tilde{m},n,\tilde{n})) \times \\
&\quad \times F^{\alpha(m_1)\dot{\alpha}(\tilde{m}_1),\beta(n_1)\dot{\beta}(\tilde{n}_1)}_{\gamma_1(k_{\alpha\alpha})\gamma_2(k_{\alpha\beta})\dot{\gamma}_1(\tilde{k}_{\alpha\alpha})\dot{\gamma}_2(\tilde{k}_{\alpha\beta}),\gamma_1(k_{\beta\alpha})\gamma_2(k_{\beta\beta})\dot{\gamma}_1(\tilde{k}_{\beta\alpha})\dot{\gamma}_2(\tilde{k}_{\beta\beta})} \times \\
&\quad \times G^{\alpha(m_2)\dot{\alpha}(\tilde{m}_2)\gamma_1(k_{\alpha\alpha}+k_{\beta\alpha})\dot{\gamma}_1(\tilde{k}_{\alpha\alpha}+\tilde{k}_{\beta\alpha}),\beta(n_2)\dot{\beta}(\tilde{n}_2)\gamma_2(k_{\alpha\beta}+k_{\beta\beta})\dot{\gamma}_2(\tilde{k}_{\alpha\beta}+\tilde{k}_{\beta\beta})} \,,
\end{aligned}
\tag{B.12}
$$

where

$$
\begin{aligned}
K(m,\tilde{m},n,\tilde{n}) \in \{ & m_1, \tilde{m}_1, n_1, \tilde{n}_1, m_2, \tilde{m}_2, n_2, \tilde{n}_2, k_{\alpha\alpha}, k_{\alpha\beta}, \\
& k_{\beta\alpha}, k_{\beta\beta}, \tilde{k}_{\alpha\alpha}, \tilde{k}_{\alpha\beta}, \tilde{k}_{\beta\alpha}, \tilde{k}_{\beta\beta} = 0,1,2,\ldots \}
\end{aligned}
\tag{B.13}
$$

with the restrictions:

$$
\begin{aligned}
m_1 + m_2 &= m \\
n_1 + n_2 &= n \\
\tilde{m}_1 + \tilde{m}_2 &= \tilde{m} \\
\tilde{n}_1 + \tilde{n}_2 &= \tilde{n}
\end{aligned}
\tag{B.14}
$$

and

$$
C(K) = \frac{i^{k_{\alpha\alpha}+k_{\beta\alpha}+\tilde{k}_{\alpha\alpha}+\tilde{k}_{\alpha\beta}-(k_{\alpha\beta}+k_{\beta\beta}+\tilde{k}_{\beta\alpha}+\tilde{k}_{\beta\beta})} m!\tilde{m}!n!\tilde{n}!}{m_1!m_2!n_1!n_2!\tilde{m}_1!\tilde{m}_2!\tilde{n}_1!\tilde{n}_2!k_{\alpha\alpha}!k_{\alpha\beta}!k_{\beta\alpha}!k_{\beta\beta}!\tilde{k}_{\alpha\alpha}!\tilde{k}_{\alpha\beta}!\tilde{k}_{\beta\alpha}!\tilde{k}_{\beta\beta}!} \,.
\tag{B.15}
$$

Of special interest is the exponential

$$\begin{aligned}\kappa(y,z) &= \exp(iy^{\alpha}z_{\alpha}) = 1 + i\epsilon^{\alpha\beta}y_{\beta}z_{\alpha} + \frac{i^2}{2!}\epsilon^{\alpha(2)\beta(2)}y_{\beta(2)}z_{\alpha(2)} + \cdots \\ \kappa^{\alpha(m),\beta(n)} &= (-i)^n n!\epsilon^{\alpha(n)\beta(n)}\delta_{m,n}\,. \end{aligned} \tag{B.16}$$

It has the property that

$$\kappa^{\alpha(m),\beta(m)}\epsilon_{\alpha(k_1)\gamma(k_1)}\epsilon_{\beta(k_2)\gamma(k_2)} = 0 \qquad \text{for} \qquad k_1 \text{ or } k_2 \geq m \qquad \text{and} \qquad k_1, k_2 > 0 \tag{B.17}$$

which greatly simplifies the calculations below.

C. Computation of $J_{\mu\nu}$

In this section we evaluate all the star products in the quantity $J_{\mu\nu}$ given in (3.18) using eqs. (B.12)–(B.15). Moreover functions of the oscillators are expanded using the convention in (B.11), except e whose expansion contains an extra factor of $i/2$, as defined in (2.13). Upon using (B.12) one obtains several different contributions corresponding to different values of the index $K(m,\tilde{m},n,\tilde{n})$ defined in (B.13). Below we shall give these quantities separately in equations labelled by A#, B#, C# and D#, where A, B, C are the contributions from the three different terms in (3.20), respectively, and D those from the first term in (3.19):

$$\begin{aligned}\mathcal{J}^{\alpha(2)}_{\mu\nu} &= \Big\{ -\mathrm{A1} + 2\times(\mathrm{B2}+\mathrm{B3}+\mathrm{B4}+\mathrm{B8}) - \\ &\quad - 2\times(\mathrm{C6}+\mathrm{C7}+\mathrm{C13}+\mathrm{C14}+\mathrm{C15}+\mathrm{C16}+\mathrm{C17}) + \mathrm{D1}\Big\} \end{aligned} \tag{C.1}$$

$$\begin{aligned}\mathcal{J}^{\alpha(1)\dot{\alpha}(1)}_{\mu\nu} &= \Big\{ -(\mathrm{A2}+\mathrm{A3}) + 2\times(\mathrm{B5}+\mathrm{B6}+\mathrm{B7}) - \\ &\quad - 2\times(\mathrm{C1}+\mathrm{C2}+\mathrm{C8}+\mathrm{C18}+\mathrm{C19}+\mathrm{C20}+\mathrm{C21}+\mathrm{C22}) + \\ &\quad + \mathrm{D3}+\mathrm{D4}\Big\} \end{aligned} \tag{C.2}$$

$$\mathcal{J}^{\dot{\alpha}(2)}_{\mu\nu} = \Big\{ -\mathrm{A4} + 2\times\mathrm{B1} - 2\times(\mathrm{C3}+\mathrm{C4}+\mathrm{C5}+\mathrm{C9}+\mathrm{C10}+\mathrm{C11}+\mathrm{C12}) + \mathrm{D2}\Big\} \tag{C.3}$$

and

$$\begin{aligned} J^{\alpha(2)}_{\mu\nu} &= \mathcal{J}^{\alpha(2)}_{\mu\nu} - \left(\mathcal{J}^{\dot{\alpha}(2)}_{\mu\nu}\right)^{\dagger}, \\ J^{\alpha\dot{\beta}}_{\mu\nu} &= \mathcal{J}^{\alpha\dot{\beta}}_{\mu\nu} - \left(\mathcal{J}^{\beta\dot{\alpha}}_{\mu\nu}\right)^{\dagger}. \end{aligned} \tag{C.4}$$

As we shall see, due to (B.17) most of the contributions obey $k_{\alpha\alpha} = k_{\alpha\beta} = k_{\beta\alpha} = 0$. In these cases we indicate by subscripts the non-trivial values of $m_1, \tilde{m}_1, n_1, \tilde{n}_1$. In the remaining cases, the subscripts indicate non-trivial values of $k_{\alpha\alpha}, k_{\alpha\beta}, k_{\beta\alpha}$. Furthermore, in listing the contributions below we split each one of them into a sum of sub-contributions A#.# etc., where the second entry labels distinct spinor index structures.

C.1 Computation of $\mathcal{L}_{\mu\nu}$

The quantity $\mathcal{L}_{\mu\nu}$, which is given in (3.20), consists of the three structures given in (3.12), (3.14) and (3.17), which are evaluated below.

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C.1.1 Evaluation of $[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]$

We have that

$$\widehat{\mathcal{L}}^{(1)}(e) = -\frac{b_1}{2} z^\alpha e_{\alpha\dot\alpha} \bar\partial^{\dot\alpha}_{(y)} \int_0^1 \int_0^1 dt' t dt \Phi(-tt'z, \bar y)\kappa(tt'z, y)\,. \tag{C.5}$$

With our conventions it follows that

$$\begin{aligned}(\widehat{\mathcal{L}}^{(1)}(e_\mu))^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} &= -\frac{b_1}{2}\frac{(-1)^{\tilde m+1}}{(n+1)n}\frac{n!}{m!(\tilde m+1)!} \times \\ &\quad \times e_{\mu,\,\dot\alpha}{}^{\beta} \Phi^{\beta(\tilde m+1)\dot\alpha(\tilde m+1)} \kappa^{\alpha(m),\beta(n-\tilde m-2)}\,, \qquad n\geq 1\,. \end{aligned}\tag{C.6}$$

Using the results of the previous section we find

$$\begin{aligned}&[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} = \\ &= b_1^2 \sum_{k_{\alpha\alpha}+k_{\beta\beta}+\tilde k_{\alpha\alpha}=odd} C(K(m,\tilde m,n,0) \times \\ &\times \frac{(-1)^{m+k_{\alpha\beta}+k_{\beta\alpha}}}{2(n_1+k_{\beta\alpha}+k_{\beta\beta}+1)(n_2+k_{\alpha\beta}+k_{\beta\beta}+1)(n_1+k_{\beta\alpha}+k_{\beta\beta})(n_2+k_{\alpha\beta}+k_{\beta\beta})} \times \\ &\times \frac{(n_1+k_{\beta\alpha}+k_{\beta\beta})!(n_2+k_{\alpha\beta}+k_{\beta\beta})!}{(m_1+k_{\alpha\alpha}+k_{\alpha\beta})!(m_2+k_{\alpha\alpha}+k_{\beta\alpha})!(\tilde m_1+\tilde k_{\alpha\alpha}+1)!(\tilde m_2+\tilde k_{\alpha\alpha}+1)!} \times \\ &\times e_{\mu,\,\dot\alpha_1}{}^{\beta_1} \Phi^{\beta_1(\tilde m_1+\tilde k_{\alpha\alpha}+1)\dot\alpha_1(\tilde m_1+\tilde k_{\alpha\alpha}+1)} \kappa^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta}),\beta_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-\tilde m_1-\tilde k_{\alpha\alpha}-2)} \times \\ &\times e_{\nu,\,\dot\alpha_2}{}^{\beta_2} \Phi^{\beta_2(\tilde m_2+\tilde k_{\alpha\alpha}+1)\dot\alpha_2(\tilde m_2+\tilde k_{\alpha\alpha}+1)} \kappa^{\alpha_2(m_2+k_{\alpha\alpha}+k_{\alpha\beta}),\beta_2(n_2+k_{\beta\alpha}+k_{\beta\beta}-\tilde m_2-\tilde k_{\alpha\alpha}-2)} \times \\ &\times \delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)} \delta^{\dot\alpha(\tilde m)}_{\dot\alpha_1(\tilde m_1)\dot\alpha_2(\tilde m_2)} \delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)} \times \\ &\times \epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})} \epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})} \epsilon_{\beta_1(k_{\beta\alpha})\alpha_2(k_{\beta\alpha})} \epsilon_{\beta_1(k_{\beta\beta})\beta_2(k_{\beta\beta})} \epsilon_{\dot\alpha_1(\tilde k_{\alpha\alpha})\dot\alpha_2(\tilde k_{\alpha\alpha})}\,. \end{aligned}\tag{C.7}$$

A1.1-2:

$$\begin{aligned}[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\dot\alpha(2)}_{\tilde m_1=1} &= -ib_1^2 \sum_{k=0}^\infty \frac{k}{(k+3)^2(k+2)^2(k+1)(k!)^2} \times \\ &\quad \times \Big\{ (\bar e_\mu e_\nu)_{\dot\gamma_2(2)} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi^{\dot\alpha(1)\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)} + \\ &\qquad + (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_{\nu,\gamma_3\dot\gamma_3} \Phi^{\dot\alpha(1)\dot\gamma_2\gamma_3}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\dot\alpha(1)\dot\gamma_3\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} \Big\}\,. \end{aligned}$$

A2.1-3:

$$\begin{aligned}&[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(1)\dot\alpha(1)}_{m_1=1,\tilde m_1=0} = \\ &= -\frac{b_1^2}{2} \sum_{k=0}^\infty \frac{k}{(k+3)^2(k+2)^2(k!)^2} \times \\ &\quad \times \Big\{ (\bar e_\mu e_\nu)_{\dot\gamma_2(2)} \Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} - \\ &\qquad - e_{\mu,\gamma_2\dot\gamma_2} e_{\nu,\dot\gamma_1}^{\alpha(1)} \Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\dot\alpha(1)\gamma_2\gamma_1(k)\dot\gamma_1(k)} + \\ &\qquad + k e_{\mu,\gamma_3\dot\gamma_2} e_{\nu,\gamma_2\dot\gamma_1} \Phi^{\dot\gamma_2\gamma_2}_{\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi^{\alpha(1)\dot\alpha(1)\gamma_3\gamma_1(k-1)\dot\gamma_1(k)} \Big\}\,. \end{aligned}$$

A3.1-3:

$$[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(1)\dot\alpha(1)}_{m_1=0,\tilde m_1=1} = -[\widehat{\mathcal{L}}^{(1)}(e_\nu), \widehat{\mathcal{L}}^{(1)}(e_\mu)]^{\alpha(1)\dot\alpha(1)}_{m_1=1,\tilde m_1=0}\,.$$

A4.1-4:

$$\begin{aligned}
&[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(2)}_{m_1=1} = \\
&= ib_1^2 \sum_{k=0}^{\infty} \frac{1}{(k+4)^2(k+3)^2(k+1)(k!)^2} \times \\
&\quad \times \Bigg\{ e^{\alpha(1)}_{\mu\dot\gamma_2} e^{\alpha(1)}_{\nu\dot\gamma_1} \Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)} \Phi^{\gamma_1(k+1)\dot\gamma_1(k+1)} - \\
&\qquad - \frac{2(k+1)}{k+2} e^{\alpha(1)}_{[\mu,\dot\gamma_2} e_{\nu],\gamma_2\dot\gamma_1} \Phi^{\dot\gamma_2\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)} \Phi^{\alpha(1)\gamma_1(k)\dot\gamma_1(k+1)} - \\
&\qquad - \frac{(k+1)k}{(k+2)} e_{\mu,\gamma_2\dot\gamma_2} e_{\nu\gamma_3\dot\gamma_3} \Phi^{\alpha(1)\dot\gamma_2\gamma_3}_{\gamma_1(k-1)\dot\gamma_1(k)} \Phi^{\alpha(1)\gamma_2\dot\gamma_3\gamma_1(k-1)\dot\gamma_1(k)} - \\
&\qquad - (k+1)(\bar e_\mu e_\nu)_{\dot\gamma_2(2)} \Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)} \Phi^{\alpha(1)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k)} \Bigg\}\,.
\end{aligned}$$

In summary we find

$$\begin{aligned}
[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\dot\alpha(2)} &= \mathrm{A1} \\
[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(1)\dot\alpha(1)} &= \mathrm{A2} + \mathrm{A3} \\
[\widehat{\mathcal{L}}^{(1)}(e_\mu), \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(2)} &= \mathrm{A4}\,.
\end{aligned} \tag{C.8}$$

C.1.2 Evaluation of $[e_\mu, \widehat{\mathcal{L}}^{(2)}(e_\nu)]$

From the definition of e and $\widehat{\mathcal{L}}^{(2)}$ one finds

$$[e_\mu, \widehat{\mathcal{L}}^{(2)}(e_\nu)] = -\frac{1}{2} e^{\beta\dot\beta}_\mu e^{\alpha\dot\alpha}_\nu \left[y_\beta \bar y_{\dot\beta}, \int_0^1 \frac{dt}{t} \bar\partial^{(y)}_{\dot\alpha} A^{(2)}_\alpha (z \to tz) \right], \tag{C.9}$$

where

$$A^{(2)}_\alpha = \frac{z_\alpha}{2} \epsilon^{\delta\gamma} \int_0^1 tdt \left[A^{(1)}_\gamma, A^{(1)}_\delta \right] (z \to tz) - z_\alpha \int_0^1 tdt \frac{ib_1}{2} \hat\Phi^{(2)} \star \kappa(z \to tz)\,. \tag{C.10}$$

Substituting (C.10) into (C.9) yields

$$\begin{aligned}
[e_\mu, \widehat{\mathcal{L}}^{(2)}(e_\nu)]^{\alpha(m)\dot\alpha(\tilde m)} &= \frac{1}{2} \epsilon^{\delta\gamma} (\bar e_{[\mu} e_{\nu]})_{\dot\alpha(2)} \left[A^{(1)}_\gamma, A^{(1)}_\delta \right]^{\alpha(m)\dot\alpha(\tilde m+2)} + \\
&+ \frac{ib_1(-1)^{m+1}(m-1)!}{4} \epsilon^{\delta\beta(1)} (\bar e_{[\mu} e_{\nu]})_{\dot\alpha(2)} \times \\
&\times \left(\Phi \star \left\{ A^{(1)}_\delta |_{\tilde y \to -\tilde y} \right\} - A^{(1)}_\delta \star \Phi \right)^{\dot\alpha(\tilde m+2),\beta(m-1)} \delta^{\alpha(m)}_{\beta(m)}\,.
\end{aligned} \tag{C.11}$$

Using

$$A^{(1)}_\alpha = -\frac{ib_1}{2} z_\alpha \int_0^1 tdt \Phi(-tz, \bar y) \kappa(tz, y) \tag{C.12}$$

and

$$A_{\gamma}^{(1)\alpha(m)\dot{\alpha}(\tilde{m}),\beta(n)} = -\frac{ib_1}{2}\frac{(-1)^{n-m-1}}{n+1}\frac{n!}{m!\tilde{m}!}\times \\ \times\Phi^{\alpha'(\tilde{m})\dot{\alpha}(\tilde{m})}\kappa^{\alpha(m),\beta'(n-\tilde{m}-1)}\delta^{\beta(n)}_{\gamma\alpha'(n-m-1)\beta'(m)} \tag{C.13}$$

we determine the first term

$$[A_{\gamma}^{(1)},A_{\delta}^{(1)}]^{\alpha(m)\dot{\alpha}(\tilde{m}),\beta(n)} = \\ = \frac{-b_1^2(-1)^{n-m}}{2}\sum_{k_{\alpha\alpha}+k_{\beta\beta}+\tilde{k}_{\alpha\alpha}=odd}\frac{C(K(m,\tilde{m},n,0))}{(n_1+k_{\beta\alpha}+k_{\beta\beta}+1)!(n_2+k_{\alpha\beta}+k_{\beta\beta}+1)!}\times \\ \times\frac{(n_1+k_{\beta\alpha}+k_{\beta\beta})!(n_2+k_{\alpha\beta}+k_{\beta\beta})!}{(m_1+k_{\alpha\alpha}+k_{\alpha\beta})!(m_2+k_{\alpha\alpha}+k_{\beta\alpha})!(\tilde{m}_1+\tilde{k}_{\alpha\alpha})!(\tilde{m}_2+\tilde{k}_{\alpha\alpha})!}\times \\ \times\Phi^{\alpha'_1(\tilde{m}_1+\tilde{k}_{\alpha\alpha})\dot{\alpha}_1(\tilde{m}_1)+\tilde{k}_{\alpha\alpha}}\kappa^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta}),\beta'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-\tilde{m}_1-\tilde{k}_{\alpha\alpha}-1)}\times \\ \times\delta^{\beta_1(n_1+k_{\beta\alpha}+k_{\beta\beta})}_{\gamma\alpha'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-(m_1+k_{\alpha\alpha}+k_{\alpha\beta})-1)\beta'_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})}\times \\ \times\Phi^{\alpha'_2(\tilde{m}_2+\tilde{k}_{\alpha\alpha})\dot{\alpha}_2(\tilde{m}_2)+\tilde{k}_{\alpha\alpha}}\kappa^{\alpha_2(m_2+k_{\alpha\alpha}+k_{\alpha\beta}),\beta'_2(n_2+k_{\beta\alpha}+k_{\beta\beta}-\tilde{m}_2-\tilde{k}_{\alpha\alpha}-1)}\times \\ \times\delta^{\beta_2(n_2+k_{\beta\alpha}+k_{\beta\beta})}_{\delta\alpha'_2(n_2+k_{\beta\alpha}+k_{\beta\beta}-(m_2+k_{\alpha\alpha}+k_{\alpha\beta})-1)\beta'_2(m_2+k_{\alpha\alpha}+k_{\alpha\beta})}\times \\ \times\delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)}\delta^{\dot{\alpha}(\tilde{m})}_{\dot{\alpha}_1(\tilde{m}_1)\dot{\alpha}_2(\tilde{m}_2)}\delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)} \\ \times\epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})}\epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})}\epsilon_{\beta_1(k_{\beta\alpha})\alpha_2(k_{\beta\alpha})}\epsilon_{\beta_1(k_{\beta\beta})\beta_2(k_{\beta\beta})}\epsilon_{\dot{\alpha}_1(\tilde{k}_{\alpha\alpha})\dot{\alpha}_2(\tilde{k}_{\alpha\alpha})}\,. \tag{C.14}$$

In the end, the results are
B1.1-2:

$$\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot{\alpha}(2)}[A_{\gamma}^{(1)},A_{\delta}^{(1)}]^{\alpha(0)\dot{\alpha}(4)}_{\tilde{m}_1=2} = \\ = \frac{1}{2}ib_1^2\sum_{k=0}^{\infty}\frac{1}{(k+4)(k+2)(k+1)(k!)^2}\times \\ \times\Bigg(2(\bar{e}_{[\mu}e_{\nu]})_{\dot{\gamma}_2(2)}\Phi^{\dot{\alpha}(1)\dot{\gamma}_2}_{\gamma_1(k+1)\dot{\gamma}_1(k-1)}\Phi^{\dot{\alpha}(1)\dot{\gamma}_2\gamma_1(k+1)\dot{\gamma}_1(k-1)}+ \\ +(\bar{e}_{[\mu}e_{\nu]})_{\dot{\gamma}_2(2)}\Phi^{\dot{\gamma}_2(2)}_{\gamma_1(k+1)\dot{\gamma}_1(k-1)}\Phi^{\dot{\alpha}(2)\gamma_1(k+1)\dot{\gamma}_1(k-1)}\Bigg)\,.$$

B2:

$$\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot{\alpha}_2(2)}[A_{\gamma}^{(1)},A_{\delta}^{(1)}]^{\alpha(2)\dot{\alpha}(2)}_{m_1=\tilde{m}_1=1} = \\ = ib_1^2\sum_{k=0}^{\infty}\frac{1}{(k+4)(k!)^2}(\bar{e}_{[\mu}e_{\nu]})_{\dot{\gamma}_2(2)}\Phi^{\alpha(1)\dot{\gamma}_2(1)}_{\gamma_1(k)\dot{\gamma}_1(k)}\Phi^{\alpha(1)\dot{\gamma}_2(1)\gamma_1(k)\dot{\gamma}_1(k)}\,.$$

B3:

$$\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot{\alpha}(2)}[A_{\gamma}^{(1)},A_{\delta}^{(1)}]^{\alpha(2)\dot{\alpha}(2)}_{m_1=2,\tilde{m}_1=0} = \\ = -\frac{ib_1^2}{2}\sum_{k=0}^{\infty}\frac{1}{(k+4)(k!)^2}(\bar{e}_{[\mu}e_{\nu]})_{\dot{\gamma}_2(2)}\Phi_{\gamma_1(k)\dot{\gamma}_1(k)}\Phi^{\alpha(2)\dot{\gamma}_2(2)\gamma_1(k)\dot{\gamma}_1(k)}\,.$$

B4:

$$\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha(2)}[A^{(1)}_\gamma, A^{(1)}_\delta]^{\alpha(2)\dot\alpha(2)}_{m_1=0,\tilde m_1=2} = -\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha(2)}\epsilon^{\delta\gamma}[A^{(1)}_\delta, A^{(1)}_\gamma]^{\alpha(2)\dot\alpha(2)}_{m_1=2,\tilde m_1=0}\,.$$

B5.1-2:

$$\begin{aligned}
&\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha(2)}[A^{(1)}_\gamma, A^{(1)}_\delta]^{\alpha(1)\dot\alpha(3)}_{m_1=1} = \\
&= \frac{b_1^2}{4}\sum_{k=0}^{\infty}\frac{k}{(k+3)(k!)^2}\Bigg(2(\bar{e}_{[\mu}e_{\nu]})_{\dot\gamma_2(2)}e^{\kappa\dot\delta}\Phi^{\dot\gamma_2(1)}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2(1)\gamma_1(k)\dot\gamma_1(k-1)} + \\
&\qquad + (\bar{e}_{[\mu}e_{\nu]})_{\dot\gamma_2(2)}e^{\kappa\dot\delta}\Phi^{\dot\alpha(1)}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi^{\alpha(1)\dot\gamma_2(2)\gamma_1(k)\dot\gamma_1(k-1)}\Bigg)\,.
\end{aligned}$$

B6.1-2:

$$\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha(2)}[A^{(1)}_\gamma, A^{(1)}_\delta]^{\alpha(1)\dot\alpha(3)}_{m_1=0} = -\frac{1}{2}\epsilon^{\delta\gamma}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha(2)}[A^{(1)}_\delta, A^{(1)}_\gamma]^{\alpha(1)\dot\alpha(3)}_{m_1=1}\,.$$

Now to the second term. Define the projections

$$P_\pm \equiv \frac{1\pm\bar\pi}{2}\,, \tag{C.15}$$

then it follows

$$\left(\Phi\star\left\{A^{(1)}_\delta|_{\tilde y\to-\tilde y}\right\} - A^{(1)}_\delta\star\Phi\right) = [\Phi, P_+A^{(1)}_\delta] - \{\Phi, P_-A^{(1)}_\delta\}\,. \tag{C.16}$$

We also have

$$\begin{aligned}
&[\Phi, P_+A^{(1)}_\delta]^{\dot\alpha(\tilde m),\beta(n)} = \\
&= 2\sum_{k_{\alpha\alpha}+\tilde k_{\alpha\alpha}=odd} C(K(0,\tilde m,n,0))\times \\
&\quad\times\Phi^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})\dot\alpha_1(m_1+\tilde k_{\alpha\alpha})}\left(P_+A^{(1)}_\delta\right)^{\alpha_2(m_2+k_{\alpha\alpha})\dot\alpha_2(\tilde m_2+\tilde k_{\alpha\alpha}),\beta_2(n+k_{\alpha\beta})}\times \\
&\quad\times\delta^{\dot\alpha(\tilde m)}_{\dot\alpha_1(\tilde m_1)\dot\alpha_2(\tilde m_2)}\delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)}\epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})}\epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})}\epsilon_{\dot\alpha_1(\tilde k_{\alpha\alpha})\dot\alpha_2(\tilde k_{\alpha\alpha})}
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
&\{\Phi, P_-A^{(1)}_\delta\}^{\dot\alpha(\tilde m),\beta(n)} = \\
&= 2\sum_{k_{\alpha\alpha}+\tilde k_{\alpha\alpha}=even} C(K(0,\tilde m,n,0))\times \\
&\quad\times\Phi^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})\dot\alpha_1(m_1+\tilde k_{\alpha\alpha})}\left(P_-A^{(1)}_\delta\right)^{\alpha_2(m_2+k_{\alpha\alpha})\dot\alpha_2(\tilde m_2+\tilde k_{\alpha\alpha}),\beta_2(n+k_{\alpha\beta})}\times \\
&\quad\times\delta^{\dot\alpha(\tilde m)}_{\dot\alpha_1(\tilde m_1)\dot\alpha_2(\tilde m_2)}\delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)}\epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})}\epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})}\epsilon_{\dot\alpha_1(\tilde k_{\alpha\alpha})\dot\alpha_2(\tilde k_{\alpha\alpha})}\,,
\end{aligned} \tag{C.18}$$

where

$$\begin{aligned}
\left(P_+A^{(1)}_\delta\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} &= \begin{cases} A^{(1),\alpha(m)\dot\alpha(\tilde m),\beta(n)}_\delta & \text{for}\quad \tilde m = \text{even} \\ 0 & \text{for}\quad \tilde m = \text{odd}\end{cases} \\
\left(P_-A^{(1)}_\delta\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} &= \begin{cases} A^{(1),\alpha(m)\dot\alpha(\tilde m),\beta(n)}_\delta & \text{for}\quad \tilde m = \text{odd} \\ 0 & \text{for}\quad \tilde m = \text{even}\,.\end{cases}
\end{aligned} \tag{C.19}$$

Altogether this amounts to
B7.1-2:

$$\frac{ib_1}{4}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha_2(2)}\epsilon^{\delta\beta(1)}\left(\Phi\star\left\{A^{(1)}_\delta|_{\bar y\to -\bar y}\right\}-A^{(1)}_\delta\star\Phi\right)^{\dot\alpha(3)}\delta^{\alpha(1)}_{\beta(1)}=$$
$$=\frac{1}{4}b_1^2\sum_{k=0}^{\infty}\frac{k}{(k!)^2}\left(2\epsilon^{\delta\alpha(1)}(\bar{e}_{[\mu}e_{\nu]})_{\dot\gamma_2(2)}\Phi^{\dot\alpha(1)\dot\gamma_2(1)}_{\delta\gamma_1(k)\dot\gamma_1(k-1)}\Phi^{\dot\gamma_2(1)\gamma_1(k)\dot\gamma_1(k-1)}+\right.$$
$$\left.+\epsilon^{\delta\alpha(1)}(\bar{e}_{[\mu}e_{\nu]})_{\dot\gamma_2(2)}\Phi^{\dot\gamma_2(2)}_{\delta\gamma_1(k)\dot\gamma_1(k-1)}\Phi^{\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)}\right).$$

B8.1-2:

$$-\frac{ib_1}{4}(\bar{e}_{[\mu}e_{\nu]})_{\dot\alpha_2(2)}\epsilon^{\delta\beta(1)}\left(\Phi\star\left\{A^{(1)}_\delta|_{\bar y\to -\bar y}\right\}-A^{(1)}_\delta\star\Phi\right)^{\dot\alpha(2),\beta(1)}\delta^{\alpha(2)}_{\beta(2)}=$$
$$=\frac{i}{2}b_1^2\sum_{k=0}^{\infty}\frac{1}{(k+3)(k!)^2}(\bar{e}_{[\mu}e_{\nu]})_{\dot\gamma_2(2)}\times$$
$$\times\left\{\epsilon^{\delta\alpha(1)}\Phi^{\dot\gamma_2(1)}_{\delta\gamma_1(k)\dot\gamma_1(k)}\Phi^{\alpha(1)\dot\gamma_2(1)\gamma_1(k)\dot\gamma_1(k)}+\epsilon^{\delta\alpha(1)}\Phi^{\alpha(1)\dot\gamma_2(2)}_{\delta\gamma_1(k)\dot\gamma_1(k)}\Phi^{\gamma_1(k)\dot\gamma_1(k)}\right\}.$$

In summary we find

$$\begin{aligned}
[e_\mu,\widehat{\mathcal{L}}^{(2)}(e_\nu)]^{\alpha(2)} &= \mathrm{B2}+\mathrm{B3}+\mathrm{B4}+\mathrm{B8}\\
[e_\mu,\widehat{\mathcal{L}}^{(2)}(e_\nu)]^{\alpha(1)\dot\alpha(1)} &= \mathrm{B5}+\mathrm{B6}+\mathrm{B7}\\
[e_\mu,\widehat{\mathcal{L}}^{(2)}(e_\nu)]^{\dot\alpha(2)} &= \mathrm{B1}\,.
\end{aligned}\tag{C.20}$$

C.1.3 Evaluation of $[e_\mu,\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_\nu)]$

From the definition of $\widehat{\mathcal{L}}^{(1)}$ one obtains

$$\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e)=-i\epsilon^{\delta\gamma}\int_0^1\frac{dt}{t}\Big(\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e)\right\}-\left[A^{(1)}_\gamma,\partial^{(y)}_\delta\widehat{\mathcal{L}}^{(1)}(e)\right]\Big)(z\to tz)\,.\tag{C.21}$$

The commutator in (3.17) is

$$[e_\mu,\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(m)\dot\alpha(\tilde m)}=-ie_{\mu,\beta\dot\alpha}\left(\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_\nu)\right)^{\alpha(m)\dot\alpha(\tilde m+1)\beta}\tag{C.22}$$

with the components

$$\left(\widehat{\mathcal{L}}^{(1)}\circ\widehat{\mathcal{L}}^{(1)}(e_\nu)\right)^{\alpha(m)\dot\alpha(\tilde m+1),\beta}=-i\epsilon^{\delta\gamma}\Big(\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e)\right\}-\left[A^{(1)}_\gamma,\partial^{(y)}_\delta\widehat{\mathcal{L}}^{(1)}(e)\right]\Big)^{\alpha(m)\dot\alpha(\tilde m+1),\beta},\tag{C.23}$$

where

$$\begin{aligned}
&\left(\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e)\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)}=\\
&=-\frac{i}{n+1}e^{\kappa\dot\delta}\left(\bar\partial^{(y)}_{\dot\delta}\partial^{(z)}_\delta A^{(1)}_\kappa\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)}\\
&=-\frac{i}{n+1}e^{\kappa\dot\delta}\epsilon_{\dot\delta\dot\alpha}\epsilon_{\delta\beta}A^{(1)\alpha(m)\dot\alpha(\tilde m+1),\beta(n+1)}_\kappa\\
&=-\frac{b_1}{2}\frac{(-1)^{n-m}}{n+2}\frac{n!}{m!(\tilde m+1)!}e^{\kappa\dot\delta}\epsilon_{\delta\beta}\Phi^{\alpha'(\tilde m+1)\dot\alpha(\tilde m)}_{\dot\delta}{}_{\kappa}{}^{\alpha(m),\beta'(n-\tilde m-1)}\delta^{\beta(n+1)}_{\kappa\alpha'(n-m)\beta'(m)}
\end{aligned}\tag{C.24}$$

and

$$
\begin{aligned}
&\left(\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e)\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} = \\
&= -\frac{i}{n}e^{\kappa\dot\delta}\left(\bar\partial_{\dot\delta}^{(y)}\partial_\delta^{(y)}A_\kappa^{(1)}\right)^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} \\
&= -\frac{i}{n}e^{\kappa\dot\delta}\epsilon_{\dot\delta\dot\alpha}\epsilon_{\delta\alpha}A_\kappa^{(1)\alpha(m+1)\dot\alpha(\tilde m+1),\beta(n)} \\
&= -\frac{b_1}{2}\frac{(-1)^{n-m}}{n+1}\frac{(n-1)!}{(m+1)!(\tilde m+1)!}e^{\kappa\dot\delta}\epsilon_{\delta\alpha}\times \\
&\quad\times\Phi_{\dot\delta}^{\alpha'(\tilde m+1)\dot\alpha(\tilde m)}{}_{\kappa}{}^{\alpha(m+1),\beta'(n-\tilde m-2)}\delta^{\beta(n)}_{\kappa\alpha'(n-m-2)\beta'(m+1)}\,.
\end{aligned}
\tag{C.25}
$$

The general expressions needed then follows as

$$
\begin{aligned}
&\left\{A_\gamma^{(1)},\partial_\delta^{(z)}\widehat{\mathcal{L}}^{(1)}(e)\right\}^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} = \\
&= \frac{ib_1^2(-1)^{n-m-1}}{2}\times \\
&\times\sum_{k_{\alpha\alpha}+k_{\beta\beta}+\tilde k_{\alpha\alpha}=\text{even}}\frac{C(K(m,\tilde m,n,0))(n_1+k_{\beta\alpha}+k_{\beta\beta})!(n_2+k_{\alpha\beta}+k_{\beta\beta})!}{(n_1+k_{\beta\alpha}+k_{\beta\beta}+1)(n_2+k_{\alpha\beta}+k_{\beta\beta}+2)}\times \\
&\times\frac{1}{(m_1+k_{\alpha\alpha}+k_{\alpha\beta})!(m_2+k_{\alpha\alpha}+k_{\beta\alpha})!(\tilde m_1+\tilde k_{\alpha\alpha})!(\tilde m_2+\tilde k_{\alpha\alpha}+1)!}\times \\
&\times e^{\kappa\dot\delta}\Phi^{\alpha'_1(\tilde m_1+\tilde k_{\alpha\alpha})\dot\alpha_1(\tilde m_1+\tilde k_{\alpha\alpha})}{}_{\kappa}{}^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta}),\beta'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-\tilde m_1-\tilde k_{\alpha\alpha}-1)}\times \\
&\times\delta^{\beta_1(n_1+k_{\beta\alpha}+k_{\beta\beta})}_{\gamma\alpha'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-m_1-k_{\alpha\alpha}-k_{\alpha\beta}-1)\beta'_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})}\times \\
&\times\epsilon_{\delta\beta_2}\Phi_{\dot\delta}^{\alpha'_2(\tilde m_2+\tilde k_{\alpha\alpha}+1)\dot\alpha_2(\tilde m_2+\tilde k_{\alpha\alpha})}{}_{\kappa}{}^{\alpha_2(m_2+k_{\alpha\alpha}+k_{\beta\alpha}),\beta'_2(n_2+k_{\alpha\beta}+k_{\beta\beta}-\tilde m_2-\tilde k_{\alpha\alpha}-1)}\times \\
&\times\delta^{\beta_2(n_2+k_{\alpha\beta}+k_{\beta\beta}+1)}_{\kappa\alpha'_2(n_2+k_{\alpha\beta}+k_{\beta\beta}-m_2-k_{\alpha\alpha}-k_{\beta\alpha})\beta'_2(m_2+k_{\alpha\alpha}+k_{\beta\alpha})}\times \\
&\times\delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)}\delta^{\dot\alpha(\tilde m)}_{\dot\alpha_1(\tilde m_1)\dot\alpha_2(\tilde m_2)}\delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)}\times \\
&\times\epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})}\epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})}\epsilon_{\beta_1(k_{\beta\alpha})\alpha_2(k_{\beta\alpha})}\epsilon_{\beta_1(k_{\beta\beta})\beta_2(k_{\beta\beta})}\epsilon_{\dot\alpha_1(\tilde k_{\alpha\alpha})\dot\alpha_2(\tilde k_{\alpha\alpha})}
\end{aligned}
\tag{C.26}
$$

and

$$
\begin{aligned}
&\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e)\right]^{\alpha(m)\dot\alpha(\tilde m),\beta(n)} = \\
&= \frac{ib_1^2(-1)^{n-m-1}}{2}\sum_{k_{\alpha\alpha}+k_{\beta\beta}+\tilde k_{\alpha\alpha}=\text{odd}}\times \\
&\times\frac{C(K(m,\tilde m,n,0))(n_1+k_{\beta\alpha}+k_{\beta\beta})!(n_2+k_{\alpha\beta}+k_{\beta\beta}-1)!}{(n_1+k_{\beta\alpha}+k_{\beta\beta}+1)(n_2+k_{\alpha\beta}+k_{\beta\beta}+1)}\times \\
&\times\frac{1}{(m_1+k_{\alpha\alpha}+k_{\alpha\beta})!(\tilde m_1+\tilde k_{\alpha\alpha})!(\tilde m_2+\tilde k_{\alpha\alpha}+1)!(m_2+k_{\alpha\alpha}+k_{\beta\alpha}+1)!}\times \\
&\times e^{\kappa\dot\delta}\Phi^{\alpha'_1(\tilde m_1+\tilde k_{\alpha\alpha})\dot\alpha_1(\tilde m_1+\tilde k_{\alpha\alpha})}{}_{\kappa}{}^{\alpha_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta}),\beta'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-\tilde m_1-\tilde k_{\alpha\alpha}-1)}\times \\
&\times\delta^{\beta_1(n_1+k_{\beta\alpha}+k_{\beta\beta})}_{\gamma\alpha'_1(n_1+k_{\beta\alpha}+k_{\beta\beta}-m_1-k_{\alpha\alpha}-k_{\alpha\beta}-1)\beta'_1(m_1+k_{\alpha\alpha}+k_{\alpha\beta})}\times \\
&\times\epsilon_{\delta\alpha_2}\Phi_{\dot\delta}^{\alpha'_2(\tilde m_2+\tilde k_{\alpha\alpha}+1)\dot\alpha(\tilde m_2+\tilde k_{\alpha\alpha})}{}_{\kappa}{}^{\alpha_2(m_2+k_{\alpha\alpha}+k_{\beta\alpha}+1),\beta'_2(n_2+k_{\alpha\beta}+k_{\beta\beta}-\tilde m_2-\tilde k_{\alpha\alpha}-2)}\times
\end{aligned}
$$

$$
\begin{aligned}
&\times\, \delta^{\beta_2(n_2+k_{\alpha\beta}+k_{\beta\beta})}_{\kappa\alpha'_2(n_2+k_{\alpha\beta}+k_{\beta\beta}-m_2-k_{\alpha\alpha}-k_{\beta\alpha}-2)\beta'_2(m_2+k_{\alpha\alpha}+k_{\beta\alpha}+1)} \times \\
&\times\, \delta^{\alpha(m)}_{\alpha_1(m_1)\alpha_2(m_2)} \delta^{\dot\alpha(\tilde m)}_{\dot\alpha_1(\tilde m_1)\dot\alpha_2(\tilde m_2)} \delta^{\beta(n)}_{\beta_1(n_1)\beta_2(n_2)} \times \\
&\times\, \epsilon_{\alpha_1(k_{\alpha\alpha})\alpha_2(k_{\alpha\alpha})} \epsilon_{\alpha_1(k_{\alpha\beta})\beta_2(k_{\alpha\beta})} \epsilon_{\beta_1(k_{\beta\alpha})\alpha_2(k_{\beta\alpha})} \epsilon_{\beta_1(k_{\beta\beta})\beta_2(k_{\beta\beta})} \epsilon_{\dot\alpha_1(\tilde k_{\alpha\alpha})\dot\alpha_2(\tilde k_{\alpha\alpha})}\,. \qquad \text{(C.27)}
\end{aligned}
$$

Projecting onto the particular components we are interested in, starting with terms not satisfying $k_{\alpha\alpha} = k_{\alpha\beta} = k_{\beta\alpha} = 0$, one finds eight anti-commutators and two commutators:
C1.1-4:

$$
\begin{aligned}
&-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma, \partial^{(z)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(1)\dot\alpha(2),\beta(1)}_{k_{\beta\alpha}=1} = \\
&= \frac{b_1^2}{4}\sum_{k=0}^\infty \frac{(k+2)k^2(k-1)}{(k+4)(k+3)^2(k!)^2} \times \\
&\quad \times e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi^{\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)}_{\dot\delta} + \frac{5b_1^2}{4}\sum_{k=0}^\infty \frac{(k+2)k}{(k+4)(k+3)^2(k!)^2} \times \\
&\quad \times \left\{ e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi^{\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)}_{\dot\delta} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \epsilon^{\alpha(1)\gamma_2} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \times \right. \\
&\qquad \left. \times\, \Phi^{\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} \right\}.
\end{aligned}
$$

C2.1-3:

$$
\begin{aligned}
&-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma, \partial^{(z)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(1)\dot\alpha(2),\beta(1)}_{k_{\beta\alpha}=2} = \\
&= \frac{b_1^2}{3}\sum_{k=0}^\infty \frac{k}{(k+4)^2(k!)^2} \times \\
&\quad \times \left\{ -k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\gamma_2\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi^{\alpha(1)\gamma_1(k-1)\dot\gamma_1(k-1)}_{\dot\delta} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \epsilon^{\alpha(1)\gamma_2} \times \right. \\
&\qquad \left. \times\, \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} - e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta} \Phi^{\gamma_2\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} \right\}.
\end{aligned}
$$

C3.1-4:

$$
\begin{aligned}
&-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma, \partial^{(z)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\dot\alpha(3),\beta(1)}_{k_{\alpha\alpha}=1} = \\
&= -\frac{ib_1^2}{2}\sum_{k=0}^\infty \frac{k}{(k+4)(k+1)(k!)^2} \times \\
&\quad \times \left\{ 2e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi^{\dot\alpha(1)\gamma_1(k+1)\dot\gamma_1(k-1)}_{\dot\delta} + 2(k+1) \times \right. \\
&\qquad \times\, e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\dot\alpha(1)\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(2)}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \times \\
&\qquad \left. \times\, \Phi^{\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)}_{\dot\delta} + (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(2)}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi^{\gamma_2\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}_{\dot\delta} \right\}.
\end{aligned}
$$

C4.1-4:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\dot\alpha(3),\beta(1)}_{k_{\alpha\beta}=k_{\beta\alpha}=1}=$$

$$=2ib_1^2\sum_{k=0}^{\infty}\frac{k}{(k+1)(k+4)^2(k!)^2}\times$$

$$\times\left\{2e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\dot\alpha(1)\gamma_1(k+1)\dot\gamma_1(k-1)}+2(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\times\right.$$

$$\times\Phi_{\dot\delta}^{\dot\alpha(1)\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\alpha(2)}_{\gamma_1(k+1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)}+$$

$$\left.+(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\dot\alpha(2)}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\dot\gamma_2\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}\right\}.$$

C5:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\dot\alpha(3),\beta(1)}_{k_{\beta\alpha}=2}=$$

$$=\frac{ib_1^2}{4}\sum_{k=0}^{\infty}\frac{k(k+5)}{(k+4)^2(k+3)(k!)^2}e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\gamma_2\dot\alpha(2)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}.$$

C6.1-5:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(2)\dot\alpha(1),\beta(1)}_{k_{\beta\alpha=1}}=$$

$$=\frac{-ib_1^2}{2}\sum_{k=0}^{\infty}\frac{(2k+7)}{(k+4)^2(k+5)(k+1)(k!)^2}\times$$

$$\times\left\{e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_1(k+1)\dot\gamma_1(k)}+\right.$$

$$+(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}+$$

$$+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_2\gamma_1(k)\dot\gamma_1(k)}+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}+$$

$$\left.+ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_2\gamma_1(k-1)\dot\gamma_1(k)}\right\}.$$

C7.1-7:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma,\partial^{(z)}_\delta\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(2)\dot\alpha(1),\beta(1)}_{k_{\beta\alpha}=2}=$$

$$=\frac{ib_1^2}{4}\sum_{k=0}^{\infty}\frac{1}{(k+5)^2(k+4)(k+2)(k+1)(k!)^2}\times$$

$$\times\left\{2(k+1)\left(e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_2\gamma_1(k)\dot\gamma_1(k)}+\right.\right.$$

$$\left.+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}\right)+$$

$$
\begin{aligned}
&+ 2(k+1)k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)} \Phi_{\dot\delta}^{\alpha(1)\gamma_2\gamma_1(k-1)\dot\gamma_1(k)} + \\
&+ (k+5)\Bigg(e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_1(k+1)\dot\gamma_1(k)} + \\
&\qquad + (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}\Bigg)\Bigg\} + \\
&+ \frac{ib_1^2}{4}\sum_{k=0}^{\infty} \frac{(k+3)}{(k+5)^2(k+4)(k+2)(k!)^2}\times \\
&\times\Bigg\{ e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta}\Phi^{\gamma_2\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)} + \\
&\quad + \frac{k(k+1)}{2} e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\Phi^{\gamma_2\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(2)\gamma_1(k-1)\dot\gamma_1(k)}\Bigg\}.
\end{aligned}
$$

C8.1-3:

$$
\begin{aligned}
&e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A^{(1)}_\gamma, \partial^{(y)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\alpha(1)\dot\alpha(2),\beta(1)}_{k_{\alpha\beta}=1} = \\
&= \frac{b_1^2}{4}\sum_{k=0}^{\infty}\frac{k}{(k+4)(k+3)(k!)^2}\Bigg\{ e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)} + \\
&\qquad + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta}\Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)} + \\
&\qquad + k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)}\Bigg\}.
\end{aligned}
$$

C9.1-4:

$$
\begin{aligned}
&e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A^{(1)}_\gamma, \partial^{(y)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\dot\alpha(3),\beta(1)}_{k_{\alpha\beta}=1} = \\
&= -\frac{ib_1^2}{2}\sum_{k=0}^{\infty}\frac{(2k+5)k}{(k+4)^2(k+3)(k+1)(k!)^2}\times \\
&\times\Bigg\{ 2e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\dot\alpha(1)\gamma_1(k+1)\dot\gamma_1(k-1)} + 2(k+1)e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\times \\
&\quad\times\Phi^{\dot\alpha(1)\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_2\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\alpha(2)}_{\gamma_1(k+1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)} + \\
&\quad + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta}\Phi^{\dot\alpha(2)}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_2\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}\Bigg\}.
\end{aligned}
$$

The remaining terms, obeying $k_{\alpha\alpha} = k_{\alpha\beta} = k_{\beta\alpha} = 0$, are (projecting first onto yy, then $\bar y\bar y$ and $y\bar y$):
C10.1-4:

$$
-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A^{(1)}_\gamma, \partial^{(z)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\dot\alpha(3),\beta(1)}_{n_1=1} =
$$

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$$= ib_1^2 \sum_{k=0}^{\infty} \frac{k}{(k+3)^2(k+2)(k+1)(k!)^2} \times$$
$$\times \Bigg(2e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\alpha(1)\gamma_1(k+1)\dot\gamma_1(k-1)} + 2(k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \times$$
$$\times \Phi_{\dot\delta}^{\dot\alpha(1)\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(2)}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)} +$$
$$+ (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(2)}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\gamma_2\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} \Bigg).$$

C11:

$$e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma} \left[A_\gamma^{(1)}, \partial_\delta^{(y)} \widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\dot\alpha(3),\beta(1)}_{n_1=1} = ib_1^2 \sum_{k=0}^{\infty} \frac{k(2k+5)}{(k+4)(k+3)(k+2)(k!)^2} \times$$
$$\times e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(2)\dot\gamma_2\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}.$$

C12.1-4:

$$e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma} \left[A_\gamma^{(1)}, \partial_\delta^{(y)} \widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\dot\alpha(3),\beta(1)}_{n_1=0} =$$
$$= ib_1^2 \sum_{k=0}^{\infty} \frac{k(2k+5)}{(k+1)(k+2)(k+3)(k!)^2} \times$$
$$\times \Bigg(2e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\alpha(1)\gamma_1(k+1)\dot\gamma_1(k-1)} + 2(k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \times$$
$$\times \Phi_{\dot\delta}^{\dot\alpha(1)\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(2)}_{\gamma_1(k+1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\gamma_2\gamma_1(k+1)\dot\gamma_1(k-1)} +$$
$$+ (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(2)}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\dot\gamma_2\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} \Bigg).$$

C13.1-5:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma} \left\{A_\gamma^{(1)}, \partial_\delta^{(z)} \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(2)\dot\alpha(1),\beta(1)}_{n_1=\tilde m_1=1} =$$
$$= ib_1^2 \sum_{k=0}^{\infty} \frac{1}{(k+4)^2(k+3)(k+1)(k!)^2} \times$$
$$\times \Bigg\{ (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)} \Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta} \epsilon^{\alpha(1)\gamma_2} \Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)} \times$$
$$\times \Phi_{\dot\delta}^{\gamma_1(k+1)\dot\gamma_1(k)} + (k+1) e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \epsilon^{\alpha(1)\gamma_2} \Phi^{\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k)} \Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)} + (k+1) \times$$
$$\times e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta} \Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)} \Phi_{\dot\delta}^{\gamma_2\gamma_1(k)\dot\gamma_1(k)} + (k+1)k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)} \times$$
$$\times \Phi_{\dot\delta}^{\alpha(1)\gamma_2\gamma_1(k-1)\dot\gamma_1(k)} \Bigg\}.$$

C14.1-y3:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma} \left\{A_\gamma^{(1)}, \partial_\delta^{(z)} \widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}^{\alpha(2)\dot\alpha(1),\beta(1)}_{n_1=1,\tilde m_1=0} =$$

$$
= -i\frac{b_1^2}{4}\sum_{k=0}^{\infty}\frac{1}{(k+4)(k+3)^2(k!)^2}\times
$$

$$
\times\left\{e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(2)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k)}+ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\times\right.
$$

$$
\left.\times\Phi_{\dot\delta}^{\alpha(2)\dot\gamma_2\gamma_2\gamma_1(k-1)\dot\gamma_1(k)}+2e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\dot\gamma_2\gamma_2\gamma_1(k)\dot\gamma_1(k)}\right\}.
$$

C15.1-4:

$$
e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\alpha(2)\dot\alpha(1),\beta(1)}_{n_1=\tilde m_1=1}=
$$

$$
=-\frac{ib_1^2}{2}\sum_{k=0}^{\infty}\frac{1}{(k+3)(k+4)(k+5)(k+1)(k!)^2}\times
$$

$$
\times\left(-(k+5)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_1(k+1)\dot\gamma_1(k)}+(k+1)^2e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\times\right.
$$

$$
\times\Phi^{\dot\gamma_2\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}-(k+1)(k+5)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\times
$$

$$
\left.\times\Phi^{\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}+3(k+1)ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\dot\gamma_2\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(2)\gamma_1(k-1)\dot\gamma_1(k)}\right).
$$

C16.1-5:

$$
e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\alpha(2)\dot\alpha(1),\beta(1)}_{n_1=0,\tilde m_1=1}
$$

$$
=ib_1^2\sum_{k=0}^{\infty}\frac{(k+3)}{(k+4)^2(k+5)(k+1)(k!)^2}\times
$$

$$
\times\left(2e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\gamma_2}_{\gamma_1(k+1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_1(k+1)\dot\gamma_1(k)}+2(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\times\right.
$$

$$
\times\Phi^{\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}+(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha\dot\delta}\Phi^{\alpha(1)\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\gamma_2\gamma_1(k)\dot\gamma_1(k)}+
$$

$$
+(k+1)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k)\dot\gamma_1(k)}+
$$

$$
\left.+(k+1)ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_2\gamma_1(k-1)\dot\gamma_1(k)}\right).
$$

C17.1-3:

$$
e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]^{\alpha(2)\dot\alpha(1),\beta(1)}_{n_1=\tilde m_1=0}=
$$

$$
=-\frac{ib_1^2}{2}\sum_{k=0}^{\infty}\frac{1}{(k+4)^2(k+5)(k+3)(k!)^2}\times
$$

$$
\times\left((2k+5)e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(1)\gamma_2\dot\gamma_2\gamma_1(k)\dot\gamma_1(k)}+\frac{2k+5}{2}e_{\mu,\gamma_2\dot\gamma_2}\times\right.
$$

$$
\times e_\nu^{\gamma_2\dot\delta}\Phi_{\gamma_1(k)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(2)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k)}+(k+2)ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\times
$$

$$\times \Phi_{\kappa\gamma_1(k-1)\dot\gamma_1(k)}\Phi_{\dot\delta}^{\alpha(2)\gamma_2\dot\gamma_2\gamma_1(k-1)\dot\gamma_1(k)}\Bigg).$$

C18.1-6:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A_\gamma^{(1)},\partial_\delta^{(z)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}_{n_1=m_1=1}^{\alpha(1)\dot\alpha(2),\beta(1)}=$$
$$=\frac{1}{2}b_1^2\sum_{k=0}^{\infty}\frac{k}{(k+3)^2(k+2)(k!)^2}\times$$
$$\times\Bigg(e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\alpha(1)\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)}+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)}\times$$
$$\times\Phi_{\dot\delta}^{\gamma_2\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)}+ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\alpha(1)\gamma_2\dot\alpha(1)\gamma_1(k-1)\dot\gamma_1(k-1)}+$$
$$+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\dot\alpha(1)}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\alpha(1)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi^{\dot\alpha(1)}_{\gamma_1(k)\dot\gamma_1(k-1)}\times$$
$$\times\Phi_{\dot\delta}^{\gamma_2\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)}+ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\dot\alpha(1)}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\alpha(1)\gamma_2\dot\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)}\Bigg).$$

C19.1-3:

$$-e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left\{A_\gamma^{(1)},\partial_\delta^{(z)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right\}_{n_1=1,m_1=0}^{\alpha(1)\dot\alpha(2),\beta(1)}=$$
$$=-\frac{b_1^2}{2}\sum_{k=0}^{\infty}\frac{k}{(k+3)^2(k+2)(k!)^2}\times$$
$$\times\Bigg(e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}+e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\gamma_2\dot\delta}\Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}+$$
$$+ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)}\Bigg).$$

C20.1-3:

$$e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]_{n_1=m_1=1}^{\alpha(1)\dot\alpha(2),\beta(1)}=$$
$$=-\frac{b_1^2}{2}\sum_{k=0}^{\infty}\frac{k}{(k+3)(k+2)(k!)^2}\times$$
$$\times\Bigg(e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\epsilon^{\alpha(1)\gamma_2}\Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}-e_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\alpha(1)\dot\delta}\Phi^{\gamma_2\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)}\times$$
$$\times\Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)}-ke_{\mu,\gamma_2\dot\gamma_2}e_\nu^{\kappa\dot\delta}\Phi^{\gamma_2\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)}\Phi_{\dot\delta}^{\alpha(1)\gamma_1(k-1)\dot\gamma_1(k-1)}\Bigg).$$

C21.1-6:

$$e_{\mu,\beta\dot\alpha}\epsilon^{\delta\gamma}\left[A_\gamma^{(1)},\partial_\delta^{(y)}\widehat{\mathcal{L}}^{(1)}(e_\nu)\right]_{n_1=0,m_1=1}^{\alpha(1)\dot\alpha(2),\beta(1)}=$$

$$
\begin{aligned}
&= \frac{1}{2} b_1^2 \sum_{k=0}^{\infty} \frac{k}{(k+3)^2 (k!)^2} \times \\
&\times \Bigg(e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\alpha(1)\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta} \Phi^{\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \times \\
&\quad \times \Phi_{\dot\delta}^{\gamma_2\dot\alpha(1)\gamma_1(k)\dot\gamma_1(k-1)} + k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\alpha(1)\gamma_2\dot\alpha(1)\gamma_1(k-1)\dot\gamma_1(k-1)} + \\
&\quad + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\dot\alpha(1)}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\alpha(1)\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\alpha(1)\dot\delta} \Phi^{\dot\alpha(1)}_{\gamma_1(k)\dot\gamma_1(k-1)} \times \\
&\quad \times \Phi_{\dot\delta}^{\gamma_2\dot\gamma_2\gamma_1(k)\dot\gamma_1(k-1)} + k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\dot\alpha(1)}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\alpha(1)\gamma_2\dot\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)} \Bigg) .
\end{aligned}
$$

C22.1-3:

$$
\begin{aligned}
&e_{\mu,\beta\dot\alpha} \epsilon^{\delta\gamma} \left[A^{(1)}_\gamma, \partial^{(y)}_\delta \widehat{\mathcal{L}}^{(1)}(e_\nu) \right]^{\alpha(1)\dot\alpha(2),\beta(1)}_{n_1=m_1=0} = \\
&= -\frac{b_1^2}{4} \sum_{k=0}^{\infty} \frac{k}{(k+3)^2 (k!)^2} \times \\
&\times \Bigg(e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\gamma_2\dot\delta} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)} + e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \epsilon^{\alpha(1)\gamma_2} \Phi^{\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\gamma_1(k)\dot\gamma_1(k-1)} + \\
&\quad + k e_{\mu,\gamma_2\dot\gamma_2} e_\nu^{\kappa\dot\delta} \Phi^{\alpha(1)\dot\alpha(1)\dot\gamma_2}_{\kappa\gamma_1(k-1)\dot\gamma_1(k-1)} \Phi_{\dot\delta}^{\gamma_2\gamma_1(k-1)\dot\gamma_1(k-1)} \Bigg) .
\end{aligned}
$$

In summary we have

$$
\begin{aligned}
[e_\mu, \widehat{\mathcal{L}}^{(1)} \circ \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(2)} &= \mathrm{C6} + \mathrm{C7} + \mathrm{C13} + \mathrm{C14} + \mathrm{C15} + \mathrm{C16} + \mathrm{C17} \\
[e_\mu, \widehat{\mathcal{L}}^{(1)} \circ \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\alpha(1)\dot\alpha(1)} &= \mathrm{C1} + \mathrm{C2} + \mathrm{C8} + \mathrm{C18} + \mathrm{C19} + \mathrm{C20} + \mathrm{C21} + \mathrm{C22} \\
[e_\mu, \widehat{\mathcal{L}}^{(1)} \circ \widehat{\mathcal{L}}^{(1)}(e_\nu)]^{\dot\alpha(2)} &= \mathrm{C3} + \mathrm{C4} + \mathrm{C5} + \mathrm{C9} + \mathrm{C10} + \mathrm{C11} + \mathrm{C12} .
\end{aligned}
\tag{C.28}
$$

C.2 Evaluation of $A^{(1)}_\gamma \star A^{(1)}_\delta$

Apart from $\mathcal{L}_{\mu\nu}$, $\mathcal{J}_{\mu\nu}$ also contain a term involving $A^{(1)}_{(\gamma} \star A^{(1)}_{\delta)}$ (see eq. (3.19)). It is convenient rewriting this as an anti-commutator, $\{A^{(1)}_\gamma, A^{(1)}_\delta\}$. The anti-commutator is evaluated using the prescription for the commutator (C.14), but replacing odd with even in the sum. The resulting structures that we are interested in read:
D1.1-3:

$$
\begin{aligned}
&i (e_{[\mu} \bar e_{\nu]})^{\gamma\delta} \{ A^{(1)}_\gamma, A^{(1)}_\delta \}^{\alpha(2)\dot\alpha(0)}_{m_1=1} = \\
&= i b_1^2 \sum_{k=0}^{\infty} \frac{1}{(k+4)^2 (k+1)^2 (k!)^2} \times \\
&\times \Bigg\{ 2(k+1) (e_{[\mu} \bar e_{\nu]})^{\gamma\delta} \delta^{\alpha(1)}_{(\gamma} \Phi_{\delta)\gamma_1(k)\dot\gamma_1(k+1)} \Phi^{\alpha(1)\gamma_1(k)\dot\gamma_1(k+1)} - (k-1)k (e_{[\mu} \bar e_{\nu]})^{\gamma\delta} \times \\
&\quad \times \Phi^{\alpha(1)}_{\gamma\gamma_1(k-1)\dot\gamma_1(k+1)} \Phi_\delta^{\alpha(1)\gamma_1(k-1)\dot\gamma_1(k)} + (e_{[\mu} \bar e_{\nu]})^{\gamma\delta} \delta^{\alpha(2)}_{\gamma\delta} \Phi_{\gamma_1(k+1)\dot\gamma_1(k+1)} \Phi^{\gamma_1(k+1)\dot\gamma_1(k+1)} \Bigg\} .
\end{aligned}
$$

D2:

$$i(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\{A^{(1)}_{\gamma}, A^{(1)}_{\delta}\}^{\alpha(0)\dot{\alpha}(2)}_{\tilde{m}_1=1} = -ib_1^2\sum_{k=0}^{\infty}\frac{1}{(k+3)^2(k!)^2}(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\Phi^{\dot{\alpha}(1)}_{\gamma\gamma_1(k)\dot{\gamma}_1(k)}\Phi_{\delta}^{\dot{\alpha}(1)\gamma_1(k)\dot{\gamma}_1(k)}\,.$$

D3.1-2:

$$\begin{aligned}
&i(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\{A^{(1)}_{\gamma}, A^{(1)}_{\delta}\}^{\alpha(1)\dot{\alpha}(1)}_{m_1=1,\tilde{m}_1=0}\\
&= -\frac{b_1^2}{2}\sum_{k=0}^{\infty}\frac{1}{(k+3)^2(k!)^2}\times\\
&\quad\times\left\{(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\delta_{\delta}^{\alpha(1)}\Phi_{\gamma_1(k)\dot{\gamma}_1(k)}\Phi_{\gamma}^{\dot{\alpha}(1)\gamma_1(k)\dot{\gamma}_1(k)} + k(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\Phi_{\delta\gamma_1(k-1)\dot{\gamma}_1(k)}\Phi_{\gamma}^{\dot{\alpha}(1)\alpha(1)\gamma_1(k-1)\dot{\gamma}_1(k)}\right\}.
\end{aligned}$$

D4.1-2:

$$i(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\{A^{(1)}_{\gamma}, A^{(1)}_{\delta}\}^{\alpha(1)\dot{\alpha}(1)}_{m_1=0,\tilde{m}_1=1} = i(e_{[\mu}\bar{e}_{\nu]})^{\gamma\delta}\{A^{(1)}_{\delta}, A^{(1)}_{\gamma}\}^{\alpha(1)\dot{\alpha}(1)}_{m_1=1,\tilde{m}_1=0}\,.$$

D. Conversion into derivatives of the scalar

The different sub-contributions A#.# idem B, C, D, listed in the previous section contain a number of different contraction patterns of the spinor indices. Below we label these patterns by a#, b#, c#, d1 or e#, and rewrite them as bilinears in derivatives of the scalar, using (2.30). Here a, b arise in the contributions from $L_{\mu\nu}$ to the Ricci tensor, c arises in the contributions from $L_{\mu\nu}$ to the contorsion, and d and e arise in the contributions from $\widehat{A}^{(1)}_{\alpha} \star \widehat{A}^{(1)}_{\beta}$ to the Ricci tensor and the contorsion, respectively. The result, which will be finally assembled in the next section, is that the contorsion tensor is made up by two distinct bilinear structures and the Ricci tensor by three distinct bilinear structures, as shown in (1.1).

D.1 Contributions from $L_{\mu\nu}$ to the Ricci tensor

The Ricci tensor $\mathrm{Ric}_{\rho\mu}$ receives contribution from $L_{\mu\nu}$ of the form $(\sigma_{\rho}{}^{\nu})_{\dot{\alpha}\dot{\beta}}L^{\dot{\alpha}\dot{\beta}}_{\mu\nu}$. Terms with an equal number of derivatives hitting the two scalars arise from the following basic structures:

$$\begin{aligned}
&\text{a1. } (\sigma_{\rho}{}^{\nu})^{\dot{\alpha}\dot{\beta}}(\sigma_{\mu})^{\beta\dot{\gamma}(1)}(\sigma_{\nu})_{\beta}{}^{\dot{\gamma}'}\Phi_{\gamma(k+1)\dot{\gamma}(k)\dot{\alpha}}\Phi^{\gamma(k+1)\dot{\gamma}(k-1)}{}_{\dot{\gamma}'\dot{\beta}} =\\
&\qquad = -2\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi \qquad\qquad \text{(D.1)}\\
&\text{a2. } (\sigma_{\rho}{}^{\nu})^{\dot{\alpha}\dot{\beta}}(\sigma_{\mu})^{\gamma'\dot{\gamma}(1)}(\sigma_{\nu})^{\gamma(1)\dot{\gamma}'}\Phi_{\gamma(k+1)\dot{\gamma}(k)\dot{\alpha}}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot{\gamma}(k-1)}{}_{\dot{\gamma}'\dot{\beta}} =\\
&\qquad = -2\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi + 4\cdot 2^k\,\nabla_{\mu\{k\}\rho}\phi\ \nabla^{\mu\{k\}}{}_{\mu}\phi \qquad\qquad \text{(D.2)}\\
&\text{a3. } (\sigma_{\rho}{}^{\nu})^{\dot{\alpha}\dot{\beta}}(\sigma_{\mu})^{\beta\dot{\gamma}(1)}(\sigma_{\nu})_{\beta}{}^{\dot{\gamma}(1)}\Phi_{\gamma(k+1)\dot{\gamma}(k+1)}\Phi^{\gamma(k+1)\dot{\gamma}(k-1)}{}_{\dot{\alpha}\dot{\beta}} =\\
&\qquad = 4\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi \qquad\qquad \text{(D.3)}\\
&\text{a4. } (\sigma_{\rho}{}^{\nu})^{\dot{\alpha}\dot{\beta}}(\sigma_{\mu})^{\gamma(1)}{}_{\dot{\alpha}}(\sigma_{\nu})^{\gamma'}{}_{\dot{\beta}}\Phi_{\gamma(k+1)\dot{\gamma}(k+1)}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot{\gamma}(k+1)} =\\
&\qquad = -6\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi \qquad\qquad \text{(D.4)}
\end{aligned}$$

$$\text{a5. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)\dot\gamma(1)}(\sigma_\nu)^{\gamma'\dot\gamma(1)}\Phi_{\gamma(k+1)\dot\gamma(k+1)}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k-1)}{}_{\dot\alpha\dot\beta} =$$
$$= -2\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi \tag{D.5}$$
$$\text{a6. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)\dot\delta}(\sigma_\nu)^{\gamma'}{}_{\dot\delta}\Phi_{\gamma(k+1)\dot\gamma(k)\dot\alpha}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k)}{}_{\dot\beta} =$$
$$= 2\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi - 8\cdot 2^k\,\nabla_{\mu\{k\}\rho}\phi\ \nabla^{\mu\{k\}}{}_\mu\phi \tag{D.6}$$
$$\text{a7. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma'\dot\gamma(1)}(\sigma_\nu)^{\gamma(1)}{}_{\dot\alpha}\Phi_{\gamma(k+1)\dot\gamma(k+1)}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k)}{}_{\dot\beta} =$$
$$= 6\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi - 4\cdot 2^k\,\nabla_{\mu\{k\}\rho}\phi\ \nabla^{\mu\{k\}}{}_\mu\phi \tag{D.7}$$
$$\text{a8. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)}{}_{\dot\alpha}(\sigma_\nu)^{\gamma'\dot\gamma'}\Phi_{\gamma(k+1)\dot\gamma(k)\dot\beta}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k)}{}_{\dot\gamma'} =$$
$$= 2\cdot 2^k\, g_{\rho\mu}\nabla_{\mu\{k+1\}}\phi\ \nabla^{\mu\{k+1\}}\phi + 4\cdot 2^k\,\nabla_{\mu\{k\}\rho}\phi\ \nabla^{\mu\{k\}}{}_\mu\phi\,, \tag{D.8}$$

where we use the notation defined in (1.2). Terms in which the number of derivatives hitting the two scalars differs by two arise from the following structures:

$$\text{b1. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)\dot\gamma(1)}(\sigma_\nu)^{\gamma(1)\dot\gamma'}\Phi_{\gamma(k+2)\dot\gamma(k)\dot\alpha\dot\beta}\Phi^{\gamma(k)\dot\gamma(k-1)}{}_{\dot\gamma'} = 4\cdot 2^k\,\nabla_{\mu\{k\}\rho\mu}\,\phi\ \nabla^{\mu\{k\}}\phi$$
$$\text{b2. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)\dot\delta}(\sigma_\nu)^{\gamma(1)}{}_{\dot\delta}\Phi_{\gamma(k+2)\dot\gamma(k)\dot\alpha\dot\beta}\Phi^{\gamma(k)\dot\gamma(k)} = -8\cdot 2^k\,\nabla_{\mu\{k\}\rho\mu}\,\phi\ \nabla^{\mu\{k\}}\phi$$
$$\text{b3. } (\sigma_\rho{}^\nu)^{\dot\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)}{}_{\dot\alpha}(\sigma_\nu)^{\gamma(1)\dot\gamma(1)}\Phi_{\gamma(k+2)\dot\gamma(k+1)\dot\beta}\Phi^{\gamma(k)\dot\gamma(k)} = 4\cdot 2^k\,\nabla_{\mu\{k\}\rho\mu}\,\phi\ \nabla^{\mu\{k\}}\phi\,. \tag{D.9}$$

There are no more structures appearing in this sector.

D.2 Contributions from $L_{\mu\nu}$ to the contorsion

The contorsion tensor receives contributions from $L_{\mu\nu}$ of the form $(\sigma^\rho)_{\alpha\dot\beta}L^{\alpha\dot\beta}_{\mu\nu}$. In all the contributions the number of derivatives hitting the two scalars differs by one, and they arise from the following structures:[1]

$$\text{c1. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\delta\dot\gamma(1)}(\sigma_\nu)_\delta{}^{\dot\gamma'}\Phi_{\gamma(k)\dot\gamma(k)}\Phi^{\gamma(k)}{}_\alpha{}^{\dot\gamma(k-1)}{}_{\dot\gamma'\dot\beta} =$$
$$= 4i\cdot 2^k\,\nabla_{\mu\{k-1\}[\mu|}\,\phi\ \nabla^{\mu\{k-1\}\rho}{}_{|\nu]}\,\phi \tag{D.10}$$
$$\text{c2. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma'\dot\gamma(1)}(\sigma_\nu)_\alpha{}^{\dot\gamma'}\Phi_{\gamma(k)\dot\gamma(k)}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k-1)}{}_{\dot\gamma'\dot\beta} =$$
$$= 2i\cdot 2^k\,\nabla_{\mu\{k-1\}[\mu|}\,\phi\ \nabla^{\mu\{k-1\}\rho}{}_{|\nu]}\,\phi - 2i\cdot 2^k g^\rho{}_{[\mu|}\,\nabla_{\mu\{k\}}\,\phi\ \nabla^{\mu\{k\}}{}_{|\nu]}\,\phi \tag{D.11}$$
$$\text{c3. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma'\dot\gamma(1)}(\sigma_\nu)^{\gamma(1)\dot\gamma'}\Phi_{\gamma(k)\dot\gamma(k)}\Phi^{\gamma(k-1)}{}_{\gamma'\alpha}{}^{\dot\gamma(k-1)}{}_{\dot\gamma'\dot\beta} = 0$$
$$\text{c4. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\delta\dot\gamma(1)'}(\sigma_\nu)_\delta{}^{\dot\gamma(1)'}\Phi_{\gamma(k)\dot\gamma(k-1)\dot\beta}\Phi^{\gamma(k)}{}_\alpha{}^{\dot\gamma(k-1)}{}_{\dot\gamma(2)'} =$$
$$= 4i\cdot 2^k\,\nabla_{\mu\{k-1\}[\mu|}\,\phi\ \nabla^{\mu\{k-1\}\rho}{}_{|\nu]} - 4i\cdot 2^k g^\rho{}_{[\mu|}\,\nabla_{\mu\{k\}}\,\phi\ \nabla^{\mu\{k\}}{}_{|\nu]}\,\phi \tag{D.12}$$
$$\text{c5. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma(1)'\dot\gamma'}(\sigma_\nu)^{\gamma(1)'\dot\gamma(1)}\Phi_{\gamma(k-1)\alpha\dot\gamma(k)}\Phi^{\gamma(k-1)}{}_{\gamma(2)'}{}^{\dot\gamma(k-1)}{}_{\dot\gamma'\dot\beta} =$$
$$= 2i\cdot 2^k\,\nabla_{\mu\{k-1\}[\mu|}\,\phi\ \nabla^{\mu\{k-1\}\rho}{}_{|\nu]}\,\phi - 2i\cdot 2^k g^\rho{}_{[\mu|}\,\nabla_{\mu\{k\}}\,\phi\ \nabla^{\mu\{k\}}{}_{|\nu]}\,\phi \tag{D.13}$$
$$\text{c6. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma'\dot\gamma'}(\sigma_\nu)_\alpha{}^{\dot\gamma(1)}\Phi_{\gamma(k)\dot\gamma(k)}\Phi^{\gamma(k)}{}_{\gamma'}{}^{\dot\gamma(k-1)}{}_{\dot\gamma'\dot\beta} =$$
$$= 2i\cdot 2^k\,\nabla_{\mu\{k-1\}[\mu|}\,\phi\ \nabla^{\mu\{k-1\}\rho}{}_{|\nu]}\,\phi + 2i\cdot 2^k g^\rho{}_{[\mu|}\,\nabla_{\mu\{k\}}\,\phi\ \nabla^{\mu\{k\}}{}_{|\nu]}\,\phi \tag{D.14}$$
$$\text{c7. } (\sigma^\rho)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma'\dot\gamma(1)'}(\sigma_\nu)_\alpha{}^{\dot\gamma(1)'}\Phi_{\gamma(k)\dot\gamma(k-1)\dot\beta}\Phi^{\gamma_k}{}_{\gamma'}{}^{\dot\gamma(k-1)}{}_{\dot\gamma(2)'} =$$

[1] Here we have omitted terms containing $\epsilon^{\mu\nu\rho\tau}$ since they are traced away in the Einstein equation

$$= \; 2i \cdot 2^k \, \nabla_{\mu\{k-1\}[\mu|} \, \phi \; \nabla^{\mu\{k-1\}\rho}{}_{|\nu]} \, \phi - 2i \cdot 2^k g^{\rho}{}_{[\mu|} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}}{}_{|\nu]} \, \phi \tag{D.15}$$

c8. $(\sigma^{\rho})^{\alpha\dot{\beta}} (\sigma_{\mu})^{\gamma' \dot{\gamma}(1)'} (\sigma_{\nu})^{\gamma(1)\dot{\gamma}(1)'} \Phi_{\gamma(k)\dot{\gamma}(k-1)\dot{\beta}} \Phi^{\gamma(k-1)}{}_{\gamma' \alpha}{}^{\dot{\gamma}(k-1)}{}_{\dot{\gamma}(2)'} =$

$$= \; 2i \cdot 2^k \, \nabla_{\mu\{k-1\}[\mu|} \, \phi \; \nabla^{\mu\{k-1\}\rho}{}_{|\nu]} \, \phi - 2i \cdot 2^k g^{\rho}{}_{[\mu|} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}}{}_{|\nu]} \, \phi \, . \tag{D.16}$$

D.3 Contributions from $\widehat{A}^{(1)}_{\alpha} \star \widehat{A}^{(1)}_{\beta}$ to Ricci tensor and contorsion

The contributions from $(\widehat{A}^{(1)}_{\alpha} \star \widehat{A}^{(1)}_{\beta})^{\dot{\alpha}\dot{\beta}}$ to the Ricci tensor contain the following structures:

d1. $(\sigma_{\mu})^{(\alpha|\dot{\alpha}} (\sigma_{\nu})^{|\beta)\dot{\beta}} \Phi_{\gamma(k)\alpha\dot{\gamma}(k)\dot{\alpha}} \Phi^{\gamma(k)}{}_{\beta}{}^{\dot{\gamma}(k)}{}_{\dot{\beta}} = -4 \cdot 2^k \, \nabla_{\mu\{k\}\mu} \, \phi \; \nabla^{\mu\{k\}}{}_{\nu} \, \phi \, +$

$$+ \, 2^k g_{\mu\nu} \, \nabla_{\mu\{k+1\}} \, \phi \; \nabla^{\mu\{k+1\}} \, \phi \, . \tag{D.17}$$

The contributions from $(\widehat{A}^{(1)}_{\alpha} \star \widehat{A}^{(1)}_{\beta})^{\dot{\alpha}\dot{\beta}}$ to the contorsion contain the following structures:

e1. $(\sigma^{\rho})_{\alpha\dot{\beta}} (\sigma_{\mu\nu})^{\delta\gamma} \epsilon_{\gamma}{}^{\alpha} \Phi_{\gamma(k)\dot{\gamma}(k)} \Phi^{\gamma(k)}{}_{\delta}{}^{\dot{\gamma}(k)\dot{\beta}} = 4i \cdot 2^k g^{\rho}{}_{[\mu|} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}}{}_{|\nu]} \, \phi$ (D.18)

e2. $(\sigma^{\rho})_{\alpha\dot{\beta}} (\sigma_{\mu\nu})^{\delta\gamma} \Phi_{\gamma(k-1)\gamma\dot{\gamma}(k)} \Phi^{\gamma(k-1)}{}_{\delta}{}^{\alpha\dot{\gamma}(k)\dot{\beta}} = 4i \cdot 2^k \, \nabla_{\mu\{k-1\}[\mu|} \, \phi \; \nabla^{\mu\{k-1\}\rho}{}_{|\nu]} \, \phi \, .$ (D.19)

D.4 Derivative of κ

In order to complete the computation of the Ricci tensor we need to differentiate the contorsion tensor. In doing so we make use of

$$\nabla \Phi_{\gamma_1 \ldots \gamma_k \dot{\gamma}_1 \ldots \dot{\gamma}_k} = i e^{\alpha\dot{\alpha}} \Phi_{\gamma_1 \ldots \gamma_k \alpha \dot{\gamma}_1 \ldots \dot{\gamma}_k \dot{\alpha}} - i k^2 e_{(\gamma_k | (\dot{\gamma}_k|} \Phi_{|\gamma_1 \ldots \gamma_{k-1})|\dot{\gamma}_1 \ldots \dot{\gamma}_{k-1})} \, , \tag{D.20}$$

which follows from (2.29). Using (2.30) we conclude that

$$\begin{aligned} \nabla_{\nu} \nabla_{\mu\{k\}} \, \phi \; &= \; \nabla_{\nu\mu\{k\}} \, \phi - \frac{k^2}{2} \, g_{\nu\{\mu_k} \nabla_{\mu\{k-1\}\}} \, \phi \\ &= \; \nabla_{\nu\mu\{k\}} \, \phi - \frac{k^2}{2} \, g_{\nu(\mu_k} \nabla_{\mu\{k-1\})} \, \phi + + \frac{k(k-1)}{4} \, g_{(\mu_k \mu_{k-1}} \nabla_{\mu\{k-2\})\nu} \, \phi \, . \end{aligned} \tag{D.21}$$

From this we can see that the content of $\nabla_{[\rho} \kappa_{\mu]\nu}{}^{\rho}$ is of the same form as the content of $i(\sigma_{\mu}{}^{\rho})_{\alpha\beta} J_{\nu\rho}{}^{\alpha\beta}$ + h.c. Indeed, since the structures c# and e# goes into $\kappa_{\mu\nu}{}^{\rho}$, we find that $\nabla_{[\rho} \kappa_{\mu]\nu}{}^{\rho}$ contains

$$\begin{aligned} &\nabla_{[\rho|} \Big(\nabla_{\mu\{k-1\}\nu} \, \phi \; \nabla^{\mu\{k\}}{}_{|\mu]}{}^{\rho} \, \phi - \nabla_{\mu\{k-1\}}{}^{\rho} \, \phi \; \nabla^{\mu\{k\}}{}_{|\mu]\nu} \, \phi + \{\text{perm. acc. to (3.8)}\} \Big) \; = \\ &\quad = \; 2 \nabla_{[\rho|} \Big(\nabla_{\mu\{k-1\}\nu} \, \phi \; \nabla^{\mu\{k\}}{}_{|\mu]}{}^{\rho} \, \phi - \nabla_{\mu\{k-1\}}{}^{\rho} \, \phi \; \nabla^{\mu\{k\}}{}_{|\mu]\nu} \, \phi \Big) \; = \\ &\quad = 2 \nabla_{\mu\{k\}\mu} \, \phi \; \nabla^{\mu\{k\}}{}_{\nu} \, \phi \; - \; k(k+2) \nabla_{\mu\{k-1\}\mu} \, \phi \; \nabla^{\mu\{k-1\}}{}_{\nu} \, \phi \; - \\ &\qquad - \; k \nabla_{\mu\{k-1\}} \, \phi \; \nabla^{\mu\{k-1\}}{}_{\mu\nu} \, \phi \; - \; \frac{k+2}{2} g_{\mu\nu} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}} \, \phi \end{aligned} \tag{D.22}$$

and

$$\begin{aligned} &\nabla_{[\rho|} \Big(g_{|\mu]\nu} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}\rho} \, \phi - g_{|\mu]}{}^{\rho} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}}{}_{\nu} \, \phi + \{\text{perm. acc. to (3.8)}\} \Big) \; = \\ &\quad = \; 2 \nabla_{[\rho|} \Big(g_{|\mu]\nu} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}\rho} \, \phi - g_{|\mu]}{}^{\rho} \, \nabla_{\mu\{k\}} \, \phi \; \nabla^{\mu\{k\}}{}_{\nu} \, \phi \Big) \; = \end{aligned}$$

$$
\begin{aligned}
&= 2\nabla_\mu(\nabla_{\mu\{k\}}\,\phi\;\nabla^{\mu\{k\}}{}_\nu\,\phi) \;+\; g_{\mu\nu}\,\nabla_\rho(\nabla_{\mu\{k\}}\,\phi\;\nabla^{\mu\{k\}\rho}\,\phi) \;= \\
&= 2\nabla_{\mu\{k\}\mu}\,\phi\;\nabla^{\mu\{k\}}{}_\nu\,\phi \;+\; 2\nabla_{\mu\{k\}}\,\phi\nabla^{\mu\{k\}}{}_{\mu\nu}\,\phi + g_{\mu\nu}\,\nabla_{\mu\{k+1\}}\,\phi\nabla^{\mu\{k+1\}}\,\phi - \\
&\quad - \frac{1}{2}(k^2+2k+2)g_{\mu\nu}\,\nabla_{\mu\{k\}}\,\phi\nabla^{\mu\{k\}}\,\phi \;- \\
&\quad - k^2\nabla_{\mu\{k-1\}\mu}\,\phi\nabla^{\mu\{k-1\}}{}_\nu\,\phi - \;k^2\nabla_{\mu\{k-1\}}\,\phi\nabla^{\mu\{k-1\}}{}_{\mu\nu}\,\phi\,. \qquad \text{(D.23)}
\end{aligned}
$$

Note that terms from $\nabla_{[\rho}\kappa_{\mu]\nu}{}^\rho$ contributes on two levels in the sum, since the number of derivatives differ among the terms in (D.22) and (D.23).

E. Computation of the Ricci tensor

In this section we use the results of section C and D to evaluate the expression for $\mathrm{Ric}_{\mu\nu} + 3g_{\mu\nu}$ given in (3.7). We divide the computation into the evaluation of the contorsion term, the $\widehat{A}^{(1)}_\alpha \star \widehat{A}^{(1)}_\beta$ term, and the $L_{\mu\nu}$ term.

E.1 The contorsion term

From (3.8) it follows that the contorsion term is given by

$$
2\nabla_{[\rho}\kappa_{\mu]\nu}{}^\rho = i\nabla_{[\rho}\Big((\sigma_{\mu]})^{\alpha\dot\beta}J_\nu{}^\rho{}_{\alpha\dot\beta} + \{\text{perm. acc. to (3.8)}\}\Big)\,. \qquad \text{(E.1)}
$$

The contributions to $J_\nu{}^\rho{}_{\alpha\dot\beta}$ are summarised in (C.2). To perform the contraction by $(\sigma_\mu)^{\alpha\dot\beta}$ we use (2.14) and expand the result in terms of the c# and e# structures defined in Section D.2 and Section D.3. We write this as

$$
\begin{aligned}
(\sigma_\mu)^{\alpha\dot\beta}\,[\mathrm{A2.1}]_\nu{}^\rho{}_{\alpha\dot\beta} &= \frac{1}{4}\sum_{k=0}^{\infty}\frac{-b_1^2 k\,[\mathrm{c1}]_{\mu\nu}{}^\rho(k)}{2(k+3)^2(k+2)^2(k!)^2} \\
&\equiv b_1^2\sum_{k=0}^{\infty}\frac{f_{\mathrm{A2.1}}(k)}{(k!)^2}\,[\mathrm{c1}]_{\mu\nu}{}^\rho(k) \qquad \text{(E.2)} \\
(\sigma_\mu)^{\alpha\dot\beta}\,[\mathrm{A2.2}]_\nu{}^\rho{}_{\alpha\dot\beta} &= -\frac{1}{4}\sum_{k=0}^{\infty}\frac{b_1^2 k\,[\mathrm{c2}]_{\mu\nu}{}^\rho(k)}{2(k+3)^2(k+2)^2(k!)^2} \\
&\equiv b_1^2\sum_{k=0}^{\infty}\frac{f_{\mathrm{A2.2}}(k)}{(k!)^2}\,[\mathrm{c2}]_{\mu\nu}{}^\rho(k) \qquad \text{(E.3)}
\end{aligned}
$$

and so on schematically as

$$
(\sigma_\mu)^{\alpha\dot\beta}\,[\mathrm{A\#.\#}]_\nu{}^\rho{}_{\alpha\dot\beta} = b_1^2\sum_{k=0}^{\infty}\frac{f_{\mathrm{A\#.\#}}(k)}{(k!)^2}\,[\mathrm{c\#'}\text{ or }\mathrm{e\#'}]_{\mu\nu}{}^\rho(k)\,, \qquad \text{(E.4)}
$$

where $f_{\mathrm{A\#.\#}}(k)$ is the coefficient of $e_\mu e_\nu\Phi\Phi$ in A#.#, which is a rational function of k which is easy to read off. An analogous notation is used for contributions of type B, C and D. All the required matches between the index structures can be found in table 1. We can now write

Table of structures	
A1.1	a1
A1.2	a2
A2.1	c1
A2.2	−c2
A2.3	c3
A3.1	c1
A3.2	−c2
A3.3	c3
A4.1	a4
A4.2	a7
A4.3	−a2
A4.4	a6
B1.1	a1
B1.2	a3
B2	a6
B3	b2
B4	b2
B5.1	c1
B5.2	c4
B6.1	c1
B6.2	c4
B7.1	c1
B7.2	c4
B8.1	a6
B8.2	b2
C1.1-2	c3
C1.3	−c2
C1.4	−c1
C2.1	c5
C2.2	−c2
C2.3	−c6
C3.1	a1
C3.2	a2
C3.3	a3
C3.4	−a5
C4.1	a1
C4.2	a2
C4.3	a3
C4.4	−a5
C5	b1
C6.1	−a4
C6.2-3	a7
C6.4	−a6
C6.5	−a2
C7.1	a7
C7.2	−a6
C7.3	−a2
C7.4	−a4
C7.5	a7
C7.6	−a8
C7.7	a5
C8.1	−c2
C8.2	−c1
C8.3	c3
C9.1	a1
C9.2	a2
C9.3	a3
C9.4	−a5
C10.1	a1
C10.2	a2
C10.3	a3
C10.4	−a5
C11	b1
C12.1	a1
C12.2	a2
C12.3	a3
C12.4	−a5
C13.1	−a6
C13.2	−a4
C13.3-4	a7
C13.5	−a2
C14.1	−b2
C14.2	−b1
C14.3	−b3
C15.1	−a4
C15.2	−a8
C15.3	a7
C15.4	a5
C16.1	−a4
C16.2-3	a7
C16.4	−a6
C16.5	−a2
C17.1	−b3
C17.2	−b2
C17.3	−b1
C18.1	c1
C18.2	c2
C18.3	−c3
C18.4	c4
C18.5	c7
C18.6	c8
C19.1	−c2
C19.2	−c1
C19.3	c3
C20.1	−c2
C20.2	−c6
C20.3	c5
C21.1	c1
C21.2	c2
C21.3	−c3
C21.4	c4
C21.5	c7
C21.6	c8
C22.1	−c1
C22.2	−c2
C22.3	c3
D2	d1
D3.1	e1
D3.2	e2
D4.1	e1
D4.2	e2

Table 1: Here we tabulate the transformation of terms of appendix C into structures of appendix D. The signs indicate that the two index patterns differ by a sign (due to flipping of spinor indices). The D1.1-3 structures only contribute to the antisymmetric part of the traced eq. (3.1). See the discussion following (3.7).

$$
\begin{aligned}
(\sigma_\mu)^{\alpha\dot\beta} J_\nu{}^\rho{}_{\alpha\dot\beta} &= b_1^2 \sum_{k=0}^{\infty} \frac{t_{ci}(k)\,[\mathrm{c}i]_{\mu\nu}{}^\rho(k) + t_{\mathrm{e}j}(k)\,[\mathrm{e}j]_{\mu\nu}{}^\rho(k)}{(k!)^2} \; - \; \text{h.c.} \\
&= 2\mathrm{Re}\{b_1^2\} \sum_{k=0}^{\infty} \frac{t_{ci}(k)\,[\mathrm{c}i]_{\mu\nu}{}^\rho(k) + t_{\mathrm{e}j}(k)\,[\mathrm{e}j]_{\mu\nu}{}^\rho(k)}{(k!)^2}\,,
\end{aligned} \tag{E.5}
$$

where summation over $i = 1, \ldots, 8$ and $j = 1, 2$ is assumed and

$$
\begin{aligned}
t_{\mathrm{c}1}(k) &= -\Big(f_{\mathrm{A}2.1}(k) + f_{\mathrm{A}3.1}(k) - 2f_{\mathrm{B}5.1}(k) - 2f_{\mathrm{B}6.1}(k) - \\
&\quad - 2f_{\mathrm{B}7.1}(k) + 2f_{\mathrm{C}1.4}(k) + 2f_{\mathrm{C}8.2}(k) + 2f_{\mathrm{C}18.1}(k) + \\
&\quad + 2f_{\mathrm{C}19.2}(k) + 2f_{\mathrm{C}21.1}(k) + 2f_{\mathrm{C}22.1}(k)\Big)\,,
\end{aligned} \tag{E.6}
$$

$$
\begin{aligned}
t_{\mathrm{c}2}(k) &= -\Big(f_{\mathrm{A}2.2}(k) + f_{\mathrm{A}3.2}(k) + 2f_{\mathrm{C}1.3}(k) + 2f_{\mathrm{C}2.2}(k) + \\
&\quad + 2f_{\mathrm{C}8.1}(k) + 2f_{\mathrm{C}18.2}(k) + 2f_{\mathrm{C}19.1}(k) + 2f_{\mathrm{C}20.1}(k) + \\
&\quad + 2f_{\mathrm{C}21.2}(k) + 2f_{\mathrm{C}22.2}(k)\Big)\,,
\end{aligned} \tag{E.7}
$$

$$
\begin{aligned}
t_{\mathrm{c}3}(k) &= -\Big(f_{\mathrm{A}2.3}(k) + f_{\mathrm{A}3.3}(k) + 2f_{\mathrm{C}1.1}(k) + 2f_{\mathrm{C}1.2}(k) + \\
&\quad + 2f_{\mathrm{C}8.3}(k) + 2f_{\mathrm{C}18.3}(k) + 2f_{\mathrm{C}19.3}(k) + 2f_{\mathrm{C}21.3}(k) + 2f_{\mathrm{C}22.3}(k)\Big)\,,
\end{aligned} \tag{E.8}
$$

$$
t_{\mathrm{c}4}(k) = -\Big(- 2f_{\mathrm{B}5.2}(k) - 2f_{\mathrm{B}6.2}(k) - 2f_{\mathrm{B}7.2}(k) + 2f_{\mathrm{C}18.4}(k) + 2f_{\mathrm{C}21.4}(k)\Big)\,, \tag{E.9}
$$

$$
t_{\mathrm{c}5}(k) = -\Big(2f_{\mathrm{C}2.1}(k) + 2f_{\mathrm{C}20.3}(k)\Big)\,, \tag{E.10}
$$

$$
t_{\mathrm{c}6}(k) = -\Big(2f_{\mathrm{C}2.3}(k) + 2f_{\mathrm{C}20.2}(k)\Big)\,, \tag{E.11}
$$

$$
t_{\mathrm{c}7}(k) = -\Big(2f_{\mathrm{C}18.5}(k) + 2f_{\mathrm{C}21.5}(k)\Big)\,, \tag{E.12}
$$

$$
t_{\mathrm{c}8}(k) = -\Big(2f_{\mathrm{C}18.6}(k) + 2f_{\mathrm{C}21.6}(k)\Big)\,, \tag{E.13}
$$

$$
t_{\mathrm{e}1}(k) = -\Big(f_{\mathrm{D}3.1}(k) + f_{\mathrm{D}4.1}(k)\Big)\,, \tag{E.14}
$$

$$
t_{\mathrm{e}2}(k) = -\Big(f_{\mathrm{D}3.2}(k) + f_{\mathrm{D}4.2}(k)\Big)\,. \tag{E.15}
$$

Finally, using (D.22) and (D.23) we compute

$$
\begin{aligned}
2\nabla_{[\rho}\kappa_{\mu]\nu}{}^\rho &= -2\mathrm{Re}\{b_1^2\} \sum_{k=0}^{\infty} \frac{2^k}{(k!)^2} \times \\
&\quad \times \Big\{ 4\Big(t_{\mathrm{c}1}(k) + t_{\mathrm{c}6}(k) + t_{\mathrm{e}1}(k) + t_{\mathrm{e}2}(k)\Big) \nabla_{\mu\{k\}\mu}\,\phi\; \nabla^{\mu\{k\}}{}_\nu\,\phi - \\
&\qquad - \Big(t_{\mathrm{c}2}(k) + 2t_{\mathrm{c}4}(k) + t_{\mathrm{c}5}(k) - t_{\mathrm{c}6}(k) + t_{\mathrm{c}7}(k) + t_{\mathrm{c}8}(k) - 2t_{\mathrm{e}1}(k)\Big) \times \\
&\qquad \times \Big(g_{\mu\nu}\,\nabla_{\mu\{k+1\}}\,\phi\; \nabla^{\mu\{k+1\}}\,\phi + 2\nabla_{\mu\{k\}}\,\phi\; \nabla^{\mu\{k\}}{}_{\mu\nu}\,\phi\Big) - \\
&\qquad - \Big(2k(k+2)(t_{\mathrm{c}1}(k) + t_{\mathrm{e}2}(k)) + 2k(k+1)t_{\mathrm{c}6}(k) + \\
&\qquad\quad + 2k(t_{\mathrm{c}2}(k) + 2t_{\mathrm{c}4}(k) + t_{\mathrm{c}5}(k) + t_{\mathrm{c}7}(k) + t_{\mathrm{c}8}(k) + k t_{\mathrm{e}1}(k))\Big) \times
\end{aligned}
$$

$$\times \nabla_{\mu\{k-1\}\mu}\, \phi\ \nabla^{\mu\{k-1\}}{}_{\nu}\, \phi\, -$$
$$- \Big((k+2)(t_{\rm c1}(k)+t_{\rm e2}(k)) + \frac{1}{2}(k^2+3k+4)t_{\rm c6}(k) + (k^2+2k+2)\times$$
$$\times\, t_{\rm e1}(k) - \frac{1}{2}k(k+1)(t_{\rm c2}(k)+2t_{\rm c4}(k)+t_{\rm c5}(k)+t_{\rm c7}(k)+t_{\rm c8}(k))\Big)\times$$
$$\times\, g_{\mu\nu}\, \nabla_{\mu\{k\}}\, \phi \nabla^{\mu\{k\}}\, \phi\, -$$
$$- \Big(2k(t_{\rm c1}(k)+kt_{\rm e1}(k)+t_{\rm e2}(k)) + k(k+1)t_{\rm c6}(k)\, -$$
$$- k(k-1)(t_{\rm c2}(k)+2t_{\rm c4}(k)+t_{\rm c5}(k)+t_{\rm c7}(k)+t_{\rm c8}(k))\Big)\times$$
$$\times \nabla_{\mu\{k-1\}}\, \phi\ \nabla^{\mu\{k-1\}}{}_{\mu\nu}\, \phi\Big\}\,. \qquad (E.16)$$

E.2 The $\widehat{A}^{(1)}_{\alpha}\star\widehat{A}^{(1)}_{\beta}$ term

In order to compute the $\widehat{A}^{(1)}_{\alpha}\star\widehat{A}^{(1)}_{\beta}$ term in (3.7) we note that

$$\begin{aligned}
&-\frac{1}{2}\Big\{(\sigma_\mu)^{\gamma\dot\alpha}(\sigma_\nu)^{\delta\dot\beta}(\widehat{A}^{(1)}_{\dot\alpha}\star\widehat{A}^{(1)}_{\dot\beta})_{\gamma\delta}\Big\}_{z=0} + \text{h.c.} = \\
&= \frac{i}{2}(\sigma_\mu{}^{\rho})^{\gamma\delta}\ 2i\Big\{e_{[\nu}{}^{\alpha\dot\alpha}e_{\rho]}{}^{\beta}{}_{\dot\alpha}(\widehat{A}^{(1)}_{\alpha}\star\widehat{A}^{(1)}_{\beta})_{\gamma\delta} + e_{[\nu}{}^{\alpha\dot\alpha}e_{\rho]\alpha}{}^{\dot\beta}(\widehat{A}^{(1)}_{\dot\alpha}\star\widehat{A}^{(1)}_{\dot\beta})_{\gamma\delta}\Big\}_{z=0} + \text{h.c.} \\
&= -\frac{i}{2}(\sigma_\mu{}^{\rho})^{\gamma\delta}\Big([\mathrm{D2}]_{\nu\rho,\dot\gamma\dot\delta}\Big)^{\dagger} + \text{h.c.} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{b_1^2\,[\mathrm{d1}]_{\mu\nu}(k)}{(k+3)^2(k!)^2} + \text{h.c.} \\
&= 2\mathrm{Re}\{b_1^2\}\sum_{k=0}^{\infty}\frac{2^k t_{\rm d1}(k)}{(k!)^2}\Big(-4\,\nabla_{\mu\{k\}\mu}\,\phi\nabla^{\mu\{k\}}{}_{\nu}\,\phi\ +\ g_{\mu\nu}\,\nabla_{\mu\{k+1\}}\,\phi\ \nabla^{\mu\{k+1\}}\,\phi\Big)\,,
\end{aligned} \qquad (E.17)$$

where $t_{\rm d1}(k) = (k+3)^{-2}/4$. (Here we do not bother to introduce $f_{\rm D2}(k)$.)

E.3 The $L_{\mu\nu}$ term

The contributions to $\mathcal{L}_{\nu\rho}{}^{\alpha\beta}$ and $\mathcal{L}_{\nu\rho}{}^{\dot\alpha\dot\beta}$ are the A, B and C terms given in (C.1) and (C.3). Contracting by $(\sigma_\mu{}^{\rho})_{\alpha\beta}$ or $(\sigma_\mu{}^{\rho})_{\dot\alpha\dot\beta}$ we get

$$\begin{aligned}
(\sigma_\mu{}^{\rho})_{\alpha\beta}\,[\mathrm{A1.1}]_{\nu\rho}{}^{\alpha\beta} &= \frac{1}{4}\sum_{k=0}^{\infty}\frac{-ib_1^2k\,[\mathrm{a1}]_{\mu\nu}(k)}{(k+3)^2(k+2)^2(k+1)(k!)^2} \\
&\equiv b_1^2\sum_{k=0}^{\infty}\frac{f_{\rm A1.1}(k)}{(k!)^2}\,[\mathrm{a1}]_{\mu\nu}(k) \qquad (E.18)\\
(\sigma_\mu{}^{\rho})_{\dot\alpha\dot\beta}\,[\mathrm{A4.1}]_{\nu\rho}{}^{\dot\alpha\dot\beta} &= \frac{1}{4}\sum_{k=0}^{\infty}\frac{ib_1^2k\,[\mathrm{a4}]_{\mu\nu}(k)}{(k+4)^2(k+3)^2(k+1)(k!)^2} \\
&\equiv b_1^2\sum_{k=0}^{\infty}\frac{f_{\rm A4.1}(k)}{(k!)^2}\,[\mathrm{a4}]_{\mu\nu}(k) \qquad (E.19)\\
(\sigma_\mu{}^{\rho})_{\alpha\beta}\,[\mathrm{C14.1}]_{\nu\rho}{}^{\alpha\beta} &= -\frac{1}{4}\sum_{k=0}^{\infty}\frac{-ib_1^2\,[\mathrm{b2}]_{\mu\nu}(k)}{4(k+4)(k+3)^2(k!)^2} \\
&\equiv b_1^2\sum_{k=0}^{\infty}\frac{f_{\rm C14.1}(k)}{(k!)^2}\,[\mathrm{b2}]_{\mu\nu}(k) \qquad (E.20)
\end{aligned}$$

and so on schematically as

$$(\sigma_\mu{}^\rho)_{\alpha\beta}\,[\mathrm{A}\#.\#]_{\nu\rho}{}^{\alpha\beta} = b_1^2 \sum_{k=0}^{\infty} \frac{f_{\mathrm{A}\#.\#}(k)}{(k!)^2}\,[\mathrm{a}\#' \text{ or } \mathrm{b}\#']_{\mu\nu}(k) \tag{E.21}$$

using the same conventions as before (see the discussion following (E.4) for details). Collecting the a# and b# contributions we have

$$\frac{i}{2}(\sigma_\mu{}^\rho)_{\alpha\beta} L_{\nu\rho}{}^{\alpha\beta} \;+\; \text{h.c.} = 2\mathrm{Re}\{b_1^2\} \sum_{k=0}^{\infty} \frac{t_{\mathrm{a}i}(k)\,[\mathrm{a}i]_{\mu\nu}(k) + t_{\mathrm{b}j}(k)\,[\mathrm{b}j]_{\mu\nu}(k)}{(k!)^2}\,, \tag{E.22}$$

where summation over $i = 1, \ldots, 8$ and $j = 1, 3$ is assumed and

$$\begin{aligned} t_{\mathrm{a}1}(k) &= -\frac{i}{2}\Big(f_{\mathrm{A}1.1}(k) - 2f_{\mathrm{B}1.1}(k) + 2f_{\mathrm{C}3.1}(k) + 2f_{\mathrm{C}4.1}(k) + \\ &\qquad + \; 2f_{\mathrm{C}9.1}(k) + 2f_{\mathrm{C}10.1}(k) + 2f_{\mathrm{C}12.1}(k)\Big)\,, \end{aligned} \tag{E.23}$$

$$\begin{aligned} t_{\mathrm{a}2}(k) &= -\frac{i}{2}\Big(f_{\mathrm{A}1.2}(k) + f_{\mathrm{A}4.3}(k) + 2f_{\mathrm{C}3.2}(k) + 2f_{\mathrm{C}4.2}(k) + \\ &\qquad + 2f_{\mathrm{C}6.5}(k) + 2f_{\mathrm{C}7.3}(k) + 2f_{\mathrm{C}9.2}(k) + 2f_{\mathrm{C}10.2}(k) + \\ &\qquad + 2f_{\mathrm{C}12.2}(k) + 2f_{\mathrm{C}13.5}(k) + f_{\mathrm{C}16.5}(k)\Big)\,, \end{aligned} \tag{E.24}$$

$$\begin{aligned} t_{\mathrm{a}3}(k) &= -\frac{i}{2}\Big(- 2f_{\mathrm{B}1.2}(k) + 2f_{\mathrm{C}3.3}(k) + 2f_{\mathrm{C}4.3}(k) + 2f_{\mathrm{C}9.3}(k) + \\ &\qquad + 2f_{\mathrm{C}10.3}(k) + 2f_{\mathrm{C}12.3}(k)\Big), \end{aligned} \tag{E.25}$$

$$\begin{aligned} t_{\mathrm{a}4}(k) &= -\frac{i}{2}\Big(f_{\mathrm{A}4.1}(k) + 2f_{\mathrm{C}6.1}(k) + 2f_{\mathrm{C}7.4}(k) + 2f_{\mathrm{C}13.2}(k) \\ &\qquad + 2f_{\mathrm{C}15.1}(k) + 2f_{\mathrm{C}16.1}(k)\Big)\,, \end{aligned} \tag{E.26}$$

$$\begin{aligned} t_{\mathrm{a}5}(k) &= -\frac{i}{2}\Big(2f_{\mathrm{C}3.4}(k) + 2f_{\mathrm{C}4.4}(k) + 2f_{\mathrm{C}7.7}(k) + 2f_{\mathrm{C}9.4}(k) + \\ &\qquad + 2\; f_{\mathrm{C}10.4}(k) + 2f_{\mathrm{C}12.4}(k) + 2f_{\mathrm{C}15.4}(k)\Big)\,, \end{aligned} \tag{E.27}$$

$$\begin{aligned} t_{\mathrm{a}6}(k) &= -\frac{i}{2}\Big(f_{\mathrm{A}4.4}(k) - 2f_{\mathrm{B}2}(k) - 2f_{\mathrm{B}8.1}(k) + 2f_{\mathrm{C}6.4}(k) + \\ &\qquad + 2f_{\mathrm{C}7.2}(k) + 2f_{\mathrm{C}13.1}(k) + 2f_{\mathrm{C}16.4}(k)\Big)\,, \end{aligned} \tag{E.28}$$

$$\begin{aligned} t_{\mathrm{a}7}(k) &= -\frac{i}{2}\Big(f_{\mathrm{A}4.2}(k) + 2f_{\mathrm{C}6.2}(k) + 2f_{\mathrm{C}6.3}(k) + 2f_{\mathrm{C}7.1}(k) + \\ &\qquad + 2f_{\mathrm{C}7.5}(k) + 2f_{\mathrm{C}13.3}(k) + 2f_{\mathrm{C}13.4}(k) + 2f_{\mathrm{C}15.3}(k) + \\ &\qquad + 2f_{\mathrm{C}16.2}(k) + 2f_{\mathrm{C}16.3}(k)\Big)\,, \end{aligned} \tag{E.29}$$

$$t_{\mathrm{a}8}(k) = -\frac{i}{2}\Big(2f_{\mathrm{C}7.6}(k) + 2f_{\mathrm{C}15.2}(k)\Big)\,, \tag{E.30}$$

$$t_{\mathrm{b}1}(k) = -\frac{i}{2}\Big(2f_{\mathrm{C}5}(k) + 2f_{\mathrm{C}11}(k) + 2f_{\mathrm{C}14.2}(k) + 2f_{\mathrm{C}17.3}(k)\Big)\,, \tag{E.31}$$

$$\begin{aligned} t_{\mathrm{b}2}(k) &= -\frac{i}{2}\Big(- 2f_{\mathrm{B}3}(k) - 2f_{\mathrm{B}4}(k) - 2f_{\mathrm{B}8.2}(k) + 2f_{\mathrm{C}14.1}(k) + \\ &\qquad + 2f_{\mathrm{C}17.2}(k)\Big)\,, \end{aligned} \tag{E.32}$$

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$$t_{b3}(k) = -\frac{i}{2}\Big(+ 2f_{C14.3}(k) + 2f_{C17.1}(k)\Big)\,. \tag{E.33}$$

From (E.22) it follows that

$$\begin{aligned}\frac{i}{2}(\sigma_\mu{}^\rho)_{\alpha\beta}L_{\nu\rho}{}^{\alpha\beta} + \text{h.c.} = \operatorname{Re}\{b_1^2\}\sum_{k=0}^\infty \frac{2^k}{(k!)^2}\Big\{ & 4\left(t_{a2}(k) - 2t_{a6}(k) - t_{a7}(k) + t_{a8}(k)\right)\times \\ & \times \nabla_{\mu\{k\}\mu}\,\phi\;\nabla^{\mu\{k\}}{}_\nu\,\phi + \\ & + 2\Big(- t_{a1}(k) - t_{a2}(k) + 2t_{a3}(k) - 3t_{a4}(k) - \\ & \quad - t_{a5}(k) + t_{a6}(k) + 3t_{a7} + t_{a8}(k)\Big)\times \\ & \times\; g_{\mu\nu}\,\nabla_{\mu\{k+1\}}\,\phi\;\nabla^{\mu\{k+1\}}\,\phi + \\ & + 4\Big(t_{b1}(k) - 2t_{b2}(k)) + t_{b3}(k)\Big)\times \\ & \times +2\nabla_{\mu\{k\}}\,\phi\;\nabla^{\mu\{k\}}{}_{\mu\nu}\,\phi\Big\}.\end{aligned} \tag{E.34}$$

E.4 The ξ, η and ζ functions

We are now ready to express $\xi(k)$, $\eta(k)$ and $\zeta(k)$, defined in (1.1), in terms of the t-functions. To this end we begin by rewriting (1.1) as

$$\begin{aligned}\text{Ric}_{\mu\nu} + 3g_{\mu\nu} = \operatorname{Re}\{b_1^2\}\Big[\sum_{k=0}^\infty \frac{2^k}{(k!)^2}\Big(&\Big(- \xi(k) - \frac{1}{2}\eta(k)\Big)\,g_{\mu\nu}\nabla_{\rho\{k+1\}}\,\phi\nabla^{\rho\{k+1\}}\,\phi + \\ & + \eta(k)\nabla_{\rho\{k\}\mu}\,\phi\;\nabla^{\rho\{k\}}{}_\nu\,\phi + \\ & + \zeta(k)\nabla_{\rho\{k\}\mu\nu}\,\phi\nabla^{\rho\{k\}}\,\phi\Big) - \frac{1}{2}\theta\;g_{\mu\nu}\,\phi\,\phi\Big],\end{aligned} \tag{E.35}$$

where the right hand side can now be identified with the sum of (E.16), (E.17) and (E.34). We find

$$\begin{aligned}-\xi(k) - \frac{1}{2}\eta(k) = & \;4\,\Big(- t_{a1}(k) - t_{a2}(k) + 2t_{a3}(k) - 3t_{a4}(k) - t_{a5}(k) + \\ & \quad + t_{a6}(k) + 3t_{a7}(k) + t_{a8}(k) + \frac{1}{2}t_{d1}(k)\Big) + \\ & + 2\,\Big(t_{c2}(k) + 2t_{c4}(k) + t_{c5}(k) - t_{c6}(k) + t_{c7}(k) + t_{c8}(k) - 2t_{e1}(k)\Big) + \\ & + 4\,\frac{k+3}{(k+1)^2}\Big(t_{c1}(k+1) + t_{e2}(k+1)\Big) - \\ & -2\frac{k+2}{k+1}\Big(t_{c2}(k+1) + 2t_{c4}(k+1) + t_{c5}(k+1) + t_{c7}(k+1) + t_{c8}(k+1)\Big) + \\ & +2\,\frac{k^2+5k+8}{(k+1)^2}t_{c6}(k+1) + \;4\,\frac{k^2+4k+5}{(k+1)^2}t_{e1}(k+1)\,,\end{aligned} \tag{E.36}$$

$$\begin{aligned}\eta(k) = & \;8\,\Big(t_{a2}(k) - 2t_{a6}(k) - t_{a7}(k) + t_{a8}(k) - t_{d1}(k)\Big) - \\ & - 8\,\Big(t_{c1}(k) + t_{c6}(k) + t_{e1}(k) + t_{e2}(k)\Big) +\end{aligned}$$

$$
\begin{aligned}
&+8\frac{k+3}{k+1}\Big(t_{\mathrm{c}1}(k+1)+t_{\mathrm{e}2}(k+1)\Big)+\\
&+8\frac{1}{k+1}\Big(t_{\mathrm{c}2}(k+1)+2t_{\mathrm{c}4}(k+1)+t_{\mathrm{c}5}(k+1)+t_{\mathrm{c}7}(k+1)+t_{\mathrm{c}8}(k+1)\Big)+\\
&+8\,\frac{k+2}{k+1}t_{\mathrm{c}6}(k+1)+8\,t_{\mathrm{e}1}(k+1)\,, \qquad \text{(E.37)}
\end{aligned}
$$

$$
\begin{aligned}
\zeta(k)\;=\;&8\,\Big(t_{\mathrm{b}1}(k)-2t_{\mathrm{b}2}(k)+t_{\mathrm{b}3}(k)\Big)+\\
&+4\,\Big(t_{\mathrm{c}2}(k)+2t_{\mathrm{c}4}(k)+t_{\mathrm{c}5}(k)-t_{\mathrm{c}6}(k)+t_{\mathrm{c}7}(k)+t_{\mathrm{c}8}(k)-2t_{\mathrm{e}1}(k)\Big)+\\
&+8\,\frac{1}{k+1}\Big(t_{\mathrm{c}1}(k+1)+t_{\mathrm{e}2}(k+1)\Big)-\\
&-4\frac{k}{k+1}\Big(t_{\mathrm{c}2}(k+1)+2t_{\mathrm{c}4}(k+1)+t_{\mathrm{c}5}(k+1)+t_{\mathrm{c}7}(k+1)+t_{\mathrm{c}8}(k+1)\Big)+\\
&+4\,\frac{k+2}{k+1}t_{\mathrm{c}6}(k+1)+8\,t_{\mathrm{e}1}(k+1)\,. \qquad \text{(E.38)}
\end{aligned}
$$

and

$$-\frac{1}{2}\theta=4t_{\mathrm{e}1}(0)\,. \qquad \text{(E.39)}$$

Upon substituting the explicit values for the t-functions given in sections E.1–E.3, we arrive at (3.22)–(3.24).

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